Introductory Calculus

Understanding the Integral

Tunc Geveci

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Introductory Calculus: Understanding the Integral

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CHAPTER 1

Using the Integral in the Approximation of Area

In this chapter, we will discuss the approximation of the area of a region between the graph of a positive-valued function and an interval.

The Summation Notation

Let's begin by introducing notation that will turn out to be convenient in expressing sums. Given numbers $a_1, a_2, \ldots, a_n$, we can indicate the sum of the numbers as

$$a_1 + a_2 + \cdots + a_n.$$

We can also indicate the sum using **the summation notation:**

$$\sum_{k=1}^{n} a_k$$

(read "sigma a_k as k runs from 1 to n"). The subscript k is **the summation index,** and is a "dummy index", in the sense that it can be replaced by any convenient letter. Thus, both

$$\sum_{j=1}^{n} a_j \text{and} \sum_{l=1}^{n} a_l$$

denote the sum $a_1 + a_2 + \cdots + a_n$.

Example 1 The sum of the first n positive integers can be expressed succinctly:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Indeed, if we set $S_n = \sum_{k=1}^{n} k$, we have

$$S_n = 1 + 2 + \cdots + (n-1) + n.$$

Let's add the terms in the opposite order:

$$S_n = n + (n-1) + \cdots + 2 + 1.$$

Therefore,

$$\begin{aligned} 2S_n &= (n+1) + ((n-1)+2) + \cdots + (2+(n-1)) + (1+n) \\ &= (n+1) + (n+1) + \ldots + (n+1) + (n+1). \end{aligned}$$

where the sum has n terms. Thus,

$$2S_n = n(n+1),$$

so that

$$S_n = \frac{n(n+1)}{2}.$$

□

The following rules are natural and easy to confirm:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k,$$

and

$$\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k \quad \text{(the constant rule for sums)}$$

Proof
By the associativity of addition,

$$\begin{aligned} &= \sum_{k=1}^{n} (a_k + b_k) = (a_1 + b_1) + \cdots + (a_n + b_n) \\ &= (a_{1+} a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n) \\ &= \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k. \end{aligned}$$

By the distributivity of multiplication with respect to sums,

$$\sum_{k=1}^{n} ca_k = ca_1 + ca_2 + \cdots + ca_n = c\left(a_1 + a_2 + \cdots + a_n\right) = c\sum_{k=1}^{n} a_k$$

□

The Area under the Graph of a Function

Assume that f is continuous on the interval $[a, b]$ and $f(x) \geq 0$ for each $x \in [a,b]$. Let G be the region in the xy-plane that is bounded by the graph of f, the interval $[a, b]$ on the x-axis, the line $x = a$ and the line $x = b$. We will refer to G simply as **the region between the graph of** f **and the interval** $[a, b]$. Our intuitive notion of the area of G is a measure of the size of G. Even though we may not be able to compute the area of G exactly, we should be able to compute approximations. We will devise a strategy that will be based on the approximation of G, in a geometric sense, by unions of rectangles.

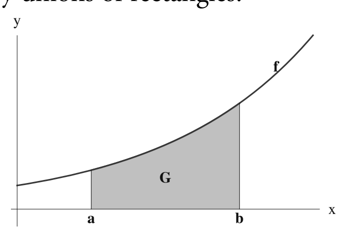


Figure 1: The region between the graph of f and the interval [a, b]

Definition 1 The set of points $P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\}$ is **a partition of the interval** $[a, b]$ if

$$a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k < \cdots < x_n = b.$$

The interval $[x_{k-1}, x_k]$ is the ***k*th subinterval** that is determined by the partition P. We will denote the length of the kth subinterval by Δx_k, so that $\Delta x_k = x_k - x_{k-1}$. The maximum of the lengths of the subintervals determined by P is the **norm of the partition** P. We will denote the norm of P by $\|P\|$, so that $\|P\|$ is the maximum of $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$. we can abbreviate the expression "maximum of $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$" as $\max_{k=1,\ldots,n} \Delta x_k$ or $\max_k \Delta x_k$. Thus,

$$\|P\| = \max_{k=1,\ldots,n} \Delta x_k .$$

Let's sample an arbitrary value of x in the kth subinterval $[x_{k-1}, x_k]$ and denote it by x_k^*.

Thus, $x_{k-1} \le x_k^* \le x_k$, but there is no other restriction on the choice of x_k^*. Consider the rectangle that has as its base the interval $[x_{k-1}, x_k]$ and has height equal to the value of f at x_k^*. If Δx_k is small, it is reasonable to approximate the area of the slice of G between the lines $x = x_{k-1}$ and $x = x_k$ by the area of such a rectangle.

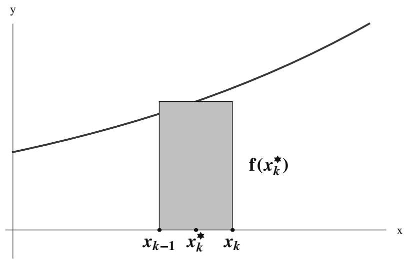


Figure 2: An approximating rectangle

The area of the rectangle is

$$f\left(x_k^*\right)\left(x_k - x_{k-1}\right) = f\left(x_k^*\right)\Delta x_k .$$

The sum of the areas of such rectangles should be a reasonable approximation to the area of G if the maximum of the lengths of the subintervals, i.e., $\|P\|$ is small:

$$\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k \cong \text{ Area of } G.$$

We would expect the approximation to be as accurate as desired if $\|P\| = \max_k \Delta x_k$ is sufficiently small.

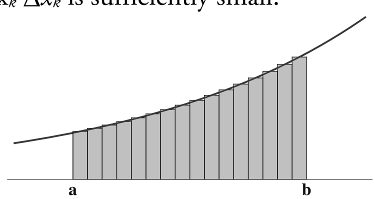


Figure 3: Approximating rectangles

Example 2 Let $f(x) = x^2 + 1$, and let G be the region between the graph of f and the interval [0, 2]. Figure 4 shows G.

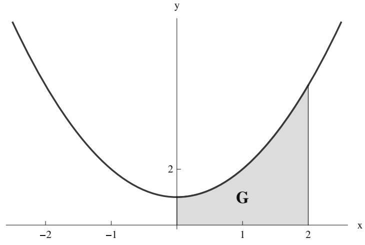


Figure 4

Let

$$P = \{0, 0.5, 1, 1.2, 1.4, 1.6, 1.8, 2\},$$

so that P is a partition of the interval [0, 2]. With reference to the notation of Definition 1, we have

$$x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.2, x_4 = 1.4, x_5 = 1.6, x_6 = 1.8 \text{ and } x_7 = 2.$$

The lengths of the subintervals determined by the partition P are

$$\Delta x_1 = \Delta x_2 = 0.5 \text{ and } \Delta x_3 = \Delta x_4 = \cdots = \Delta x_7 = 0.2.$$

Therefore, the norm of P is 0.5:

$$\|P\| = 0.5.$$

Let's form the rectangle of height $f(c_k)$ on the kth subinterval $[x_{k-1}, x_k]$, where c_k is the midpoint of $[x_{k-1}, x_k]$, $k = 1, 2, \ldots, 7$, and approximate the area of the region G by the sum of these rectangles. Figure 5 indicates the rectangles.

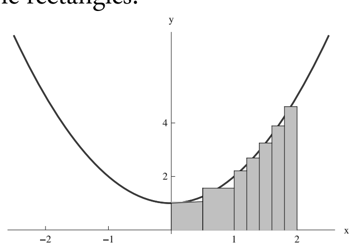


Figure 5

The approximation to the area of G is

$$\sum_{k=1}^{7} f(c_k)\Delta x_k \cong f(0.25)(0.5)+f(0.75)(0.5)+f(1.1)(0.2)$$
$$+f(1.3)(0.2)+f(1.5)(0.2)+f(1.7)(0.2)+f(1.8)(0.2)$$
$$\cong 4.5685.$$

In Section 5.4 we will show that the area of G is

$$\frac{14}{3} \cong 4.666\,67,$$

so that the absolute error of our approximation is about 0.1. For many purposes, the magnitude of the error may be unacceptable. On the other hand, we would expect the error to be as small as desired if the interval [0, 2] is partitioned to subintervals of sufficiently small length. □

In the other examples of this chapter, we will consider the partitioning of an interval $[a, b]$ into n subintervals of equal length, since the corresponding sums can be expressed and computed easily. Thus,

$$\Delta x_k = \Delta x = \frac{b-a}{n} \text{ for } k = 1, 2, \ldots, n.$$

and

$$x_k = a + k\Delta x,\ k = 0, 1, 2, \ldots, n.$$

We will approximate the area of the region between the graph of f and the interval $[a, b]$ by sums of the form

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n} f(x_k^*)\Delta x = \Delta x \sum_{k=1}^{n} f(x_k^*).$$

The intermediate points x_k^*, $k = 1, 2, \ldots, n$, can be chosen in many different ways. We will consider the following strategies:

1. A **left-endpoint sum** is obtained by choosing x_k^* to be the left endpoint x_{k-1} of the kth subinterval $[x_{k-1}, x_k]$. We have

$$x_{k-1} = a + (k-1)\Delta x.$$

We will denote the left-endpoint sum corresponding to the function f and the partitioning of the interval $[a, b]$ to n subintervals of equal length as l_n. Thus,

$$l_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x.$$

2. A **right-endpoint sum** is obtained by choosing x_k^* to be the right endpoint x_k of the kth subinterval $[x_{k-1}, x_k]$. We have

$$x_k = a + k\Delta x.$$

We will denote the right-endpoint sum corresponding to the function f and the partitioning of the interval $[a, b]$ to n subintervals of equal length as r_n. Thus,

$$r_n = \sum_{k=1}^{n} f(x_k)\Delta x.$$

3. A **midpoint sum** is obtained by choosing x_k^* to be the midpoint c_k of the kth subinterval $[x_{k-1}, x_k]$. We have

$$c_k = \frac{x_{k-1} + x_k}{2} = \frac{1}{2}(a + (k-1)\,\Delta x + a + k\Delta x)$$
$$= \frac{1}{2}(2a + (2k-1)\,\Delta x) = a + (k - \frac{1}{2})\Delta x.$$

We will denote the midpoint sum corresponding to the function f and the partitioning of the interval $[a, b]$ to n subintervals of equal length as m_n. Thus,

$$m_n = \sum_{k=1}^{n} f(c_k)\Delta x.$$

As we will discuss in more detail in the next chapter, **any of the above sums approximates the area of the region between the graph of f and the interval $[a, b]$ as accurately as desired, provided that f is continuous on $[a, b]$ and Δx is small enough.** Since

$$\Delta x = \frac{b-a}{n},$$

Δx is as small as necessary if n is sufficiently large. Therefore, **the area $A(G)$ of the region G between the graph of f and the interval $[a, b]$ is the limit of left-endpoint sums, right-endpoint sums or midpoint sums as n tends to infinity:**

$$A(G) = \lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n = \lim_{n\to\infty} m_n.$$

Example 3 Let $f(x) = x$. The region G between the graph of f and the interval $[0, 1]$ is a triangle whose base has length 1 and whose height is 1. Therefore, the area of G is

$$\frac{1}{2}(1)(1) = \frac{1}{2}.$$

Consider the approximation of the area of G by right-endpoint sums r_n. Figure 6 illustrates the rectangles that correspond to $n = 16$. Show that $\lim_{n\to\infty} r_n =$ area of G.

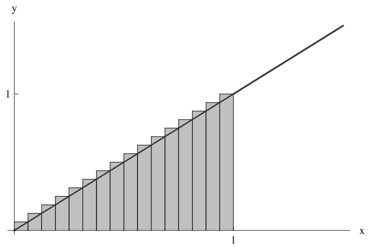


Figure 6

Solution

We have

$$r_n = \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} x_k \Delta x,$$

where

$$\Delta x = \frac{1}{n} \text{ and } x_k = k\Delta x = \frac{k}{n}.$$

Therefore,

$$r_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)\left(\frac{1}{\text{n}}\right) = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k.$$

In Example 1 we showed that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Therefore,

$$r_n = \frac{1}{n^2}\sum_{k=1}^{k} k = \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{n(n+1)}{2n^2}.$$

Thus,

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{2n^2} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2}.$$

Therefore, the area of G is 1/2. □

Example 4 Let $f(x) = x^2$.

a) Sketch the region G between the graph of f and the interval [1, 3].

b) Determine the area of G as the limit of left-endpoint sums. The following expression for the sum of the squares of the fist n positive integers will be helpful:

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

(as you can confirm by mathematical induction).

Solution

a) Figure 7 shows the graph of *f* and the region *G*.

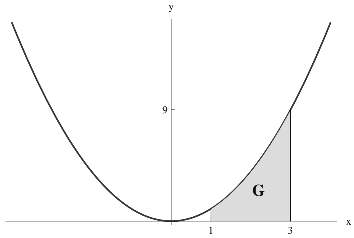


Figure 7

b) The interval [1, 3] is subdivided into *n* subintervals of length

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}.$$

The corresponding partition consists of the points

$$x_k = 1 + k\Delta x = 1 + k\left(\frac{2}{n}\right), k = 0, 1, 2, \ldots, n,$$

Therefore, the corresponding left-endpoint sum for *f* is

$$\begin{aligned} l_n = \sum_{k=1}^{n} f(x_{k-1})\Delta x = \sum_{k=1}^{n} f\left(1 + (k-1)\left(\frac{2}{n}\right)\right)\frac{2}{n} &= \frac{2}{n}\sum_{k=1}^{n}\left(1 + \frac{2(k-1)}{n}\right)^2 \\ &= \frac{2}{n}\sum_{k=1}^{n}\left(1 + \frac{4(k-1)}{n} + \frac{4(k-1)^2}{n^2}\right) \\ &= \frac{2}{n}\left(\sum_{k=1}^{n}1 + \frac{4}{n}\sum_{k=1}^{n}(k-1) + \frac{4}{n^2}\sum_{k=1}^{n}(k-1)^2\right) \\ &= \frac{2}{n}\sum_{k=1}^{n}1 + \frac{8}{n^2}\sum_{k=1}^{n}(k-1) + \frac{8}{n^3}\sum_{k=1}^{n}(k-1)^2. \end{aligned}$$

We have

$$\sum_{k=1}^{n} 1 = 1 + 1 + \cdots + 1 = n,$$

since n terms are added.

We also have

$$\sum_{k=1}^{n}(k-1) = 0+1+2+\cdots+(n-1) = \sum_{j=1}^{n-1} j = \frac{(n-1)n}{2},$$

as in Example 1 (with n replaced by $n - 1$).

Finally,

$$\sum_{k=1}^{n}(k-1)^2 = 0+1^2+3^2+\cdots+(n-1)^2 = \sum_{j=1}^{n-1} j^2.$$

We will apply the formula

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1),$$

with n replaced by $n - 1$. Thus,

$$\sum_{j=1}^{n-1} j^2 = \frac{1}{6}(n-1)(n)(2n-1).$$

Therefore,

$$\begin{aligned} l_n &= \frac{2}{n}\sum_{k=1}^{n} 1 + \frac{8}{n^2}\sum_{k=1}^{n}(k-1) + \frac{8}{n^3}\sum_{k=1}^{n}(k-1)^2 \\ &= \frac{2}{n}(n) + \frac{8}{n^2}\left(\frac{(n-1)n}{2}\right) + \frac{8}{n^3}\left(\frac{1}{6}(n-1)(n)(2n-1)\right) \\ &= 2 + \frac{4(n-1)}{n} + \frac{4(n-1)(2n-1)}{3n^2}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} l_n = 2 + 4\lim_{n\to\infty}\frac{n-1}{n} + \frac{4}{3}\lim_{n\to\infty}\frac{(n-1)(2n-1)}{n^2} = 2+4+\frac{4}{3}(2) = \frac{26}{3}.$$

Therefore, the area of the region G between the graph of f and the interval [1, 3] is 26/3.

Figure 8 shows the rectangles corresponding to the partitioning of the interval [1, 2] into 10 subintervals of equal length. □

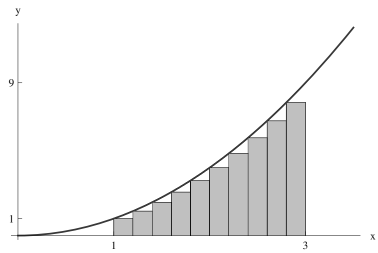


Figure 8

Example 5 Let $f(x) = \sin(x)$. In Section 5.3 we will show that the area of the region G between the graph of f and the interval $[0, \pi]$ is 2.

a) Sketch the region G.

b) Midpoint sums are usually more accurate in approximating the area, compared to left-endpoint sums and right-endpoint sums. Approximate the area of G by midpoint sums that correspond to the partitioning of $[0, \pi]$ to 2^k subintervals of equal length, where $k = 2, \ldots, 7$. Do the numbers support the expectation that it should be possible to approximate the area of G with desired accuracy by a midpoint sum, provided that the length of each subinterval is small enough?

Solution

a) Figure 9 shows the region G.

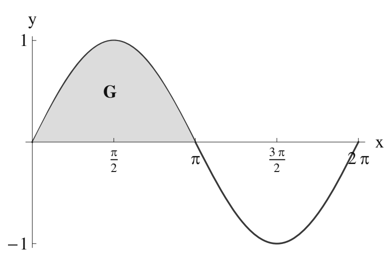


Figure 9

b) We have

$$m_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} \sin(c_k)\,\Delta x,$$

where

$$\Delta x = \frac{\pi}{n} \text{ and } c_k = \left(k - \frac{1}{2}\right)\Delta x.$$

Figure 10 shows the rectangles corresponding to a partitioning of the interval $[0, \pi]$ to 16 subintervals of equal length.

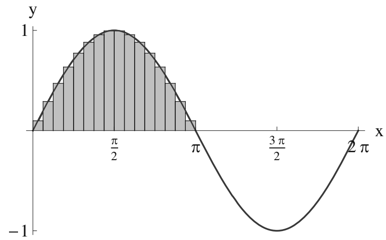


Figure 10

Table 1 displays the relevant data. The numbers support the expectation that it should be possible to approximate the area of G with desired accuracy by a midpoint sum, if the length of each subinterval is small enough. □

n	Δx	m_n	$\lvert m_n - 2 \rvert$
4	.25	2.052 34	5.2×10^{-2}
8	.125	2.012 91	1.3×10^{-2}
16	.0625	2.003 22	3.2×10^{-3}
32	.03125	2.000 8	8.0×10^{-4}
64	.015625	2.000 2	2.0×10^{-4}
128	7.8125×10^{-3}	2.000 05	5.0×10^{-5}

Table 1

In the next chapter, we will introduce a fundamental concept of calculus, namely **the integral.** You will see that the integral of a positive-valued function can be interpreted as area.

CHAPTER 2

Understanding the Concept of the Integral

In this chapter, we will introduce the fundamental concept of the **integral.** The integral of a positive-valued function on an interval is the area of the region between the graph of the function and the interval. We will be able to interpret the integral of a function that has positive or negative values on an interval as "**the signed area**" of the region between the graph of the function and the interval. In the next chapter, you will see that the **displacement** of an object in one-dimensional motion over a time interval is **the integral of the velocity function** on that interval. In later chapters, the integral will appear as the **work** done in moving an object, or as the **probability** that the values of a random variable are in a certain interval.

The Riemann Integral and Signed Area

As in Section 5.1, let $P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\}$ be a **partition** of the interval $[a, b]$, so that

$$a = x_o < x_1 < x_2 < \cdots x_{k-1} < x_k < \cdots < x_{n-1} < x_n = b.$$

Recall that $\|P\|$, **the norm of the partition** P, is the maximum of the lengths of subintervals determined by P:

$$\| P \| = \max_{k=1,\ldots,n} \Delta x_k = \max_{k=1,\ldots,n} \left(x_k - x_{k-1} \right).$$

Let f be a function defined on $[a, b]$. As in Section 5.1, we will sample an arbitrary value of x in the kth subinterval $[x_{k-1}, x_k]$ and denote it by x_k^*. Thus, $x_{k-1} \le x_k^* \le x_k$, but there is no other restriction on the choice of x_k^*.

Definition 1 Assume that $P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_n\}$ is a partition of the interval $[a, b]$, and $x_k^* \in [x_{k-1}, x_k]$. A sum of the form

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

is **a Riemann sum for** f **on the interval** $[a, b]$.

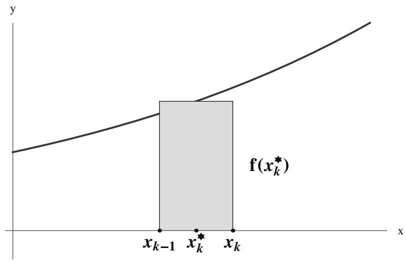


Figure 1: A typical term of a Riemann sum is $f\left(x_k^*\right) \Delta x_k$

Thus, a Riemann sum for f on $[a, b]$ approximates the area of the region between the graph of f and the interval $[a, b]$ if $f(x) \geq 0$ for each $x \in [a, b]$ and the norm of the partition is small. Let's lift the restriction on the sign of f, and **assume that any Riemann sum for** f **on** $[a, b]$ **approximates a number which depends only on the function** f **and the interval** $[a, b]$ **if the norm of the partition is small.** We will denote that number as

$$\int_a^b f(x)\,dx$$

and refer to it as **the Riemann integral of** f **on** $[a, b]$**.** You can imagine that we have replaced the summation symbol in the expression

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

by an elongated *S*, and Δx_k by *dx* (*"dx"* within the present context should not be confused with *"dx"* within the context of the differential, although a connection will arise later). We will also assume that the approximation is as accurate as desired provided that the norm of the partition is small enough. Thus, we can define the Riemann integral of f on $[a, b]$ as follows:

Definition 2 (The informal definition of the integral) We say that a function ***f*** **is Riemann integrable on the interval** [***a, b***] and that **the Riemann integral of *f* on** [***a, b***] is

$$\int_a^b f(x)\,dx$$

if

$$\left|\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k - \int_a^b f(x)\,dx\right|$$

is as small as desired provided that the norm of the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ is sufficiently small.

Thus, the Riemann integral of *f* on $[a, b]$ corresponds to the area of the region between the graph of f and $[a, b]$ if f is positive-valued on $[a, b]$.

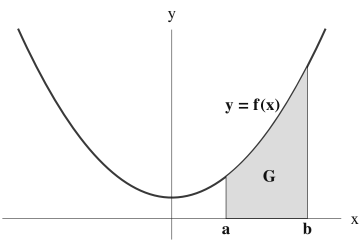


Figure 2: $\int_a^b f(x)\,dx$, ***is the area of G if f is positive-valued***

We may express the relationship between Riemann sums and the Riemann integral by writing

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k = \int_a^b f(x)\,dx.$$

You can find the precise definition of the Riemann integral at the end of this chapter. **Riemann** was a mathematician who made crucial contributions in many areas of mathematics, and played a prominent role in establishing firm foundations for the concept of the integral. Since we will not have occasion to use any other type of integral in this book, we will refer to the Riemann integral simply as “the integral”.

In the notation,

$$\int_a^b f(x)\,dx,$$

for the integral of f on $[a, b]$, the number a is referred to as **the lower limit** of the integral, and b as **the upper limit** of the integral. The function f is **the integrand.** The computation of the integral may be described by saying that "f **is integrated from** a **to** b".

We will calculate many integrals in the following chapters. Let's determine the integrals of constant functions before we proceed further. If f is constant and has the value $c > 0$, the region between the graph of f and an interval $[a, b]$ is a rectangle with area $c(b - a)$. Therefore, we should have

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

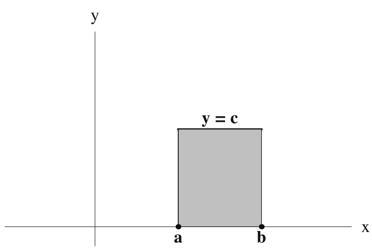


Figure 3

This is the case, irrespective of the sign of c. Indeed, for any partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and any choice of the intermediate points x_k^*,

$$\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k = \sum_{k=1}^{n} c\Delta x_k = c\sum_{k=1}^{n} \Delta x_k = c(b-a),$$

since the sum of the lengths of the subintervals is the length of the interval $[a, b]$. Let's record this fact:

Proposition 1 Let f be a constant function, so that $f(x) = c$ for each $x \in \mathbb{R}$, where c is a constant. Then

$$\int_a^b f(x)\,dx = \int_a^b c\,dx = c(b-a).$$

You can find an example of a function that is not Riemann integrable at the end of this chapter. We have the assurance every continuous function is Riemann integrable:

Theorem 1 Assume that *f* is continuous on the interval [*a, b*]. Then *f* is Riemann integrable on [*a, b*].

The proof of the theorem is left to a course in advanced calculus.

By Theorem 1, a Riemann sum

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

for the function *f* on the interval [*a, b*] approximates the integral of *f* on [*a, b*] as accurately as desired, provided that *f* is continuous on [*a, b*] and $\max_k \Delta x_k$ is small enough. In particular, we can approximate an integral by **left-endpoint sums, right-end point sums** or **midpoint sums,** as in Section 5.1 (without the restriction that the functions are positive-valued). If

$$\Delta x = \frac{b-a}{n},$$

Δx is as small as necessary if n is sufficiently large. Therefore,

$$\lim_{n\to\infty} l_n = \lim_{n\to\infty} r_n = \lim_{n\to\infty} m_n = \int_a^b f(x)\,dx,$$

with the notation of Section 5.1.

Example 1 Let $f(x) = x$, as in Example 3 of Section 5.1. In that example, we approximated the area of the region G between the graph of *f* and the interval [0, 1] by right-endpoint sums. We showed that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}.$$

Therefore,

$$\int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$

□

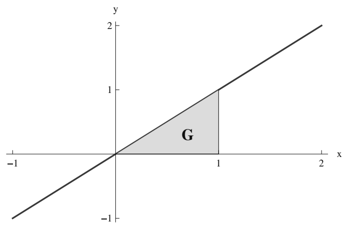


Figure 4: The area of G is $\int_0^1 x dx = 0.5$

Let's consider the case of a function f that is continuous on an interval $[a, b]$ and $f(x) \leq 0$ for each $x \in [a, b]$, and interpret the integral of f on $[a, b]$ geometrically. Let G be the region between the graph of f and $[a, b]$, as illustrated in Figure 5.

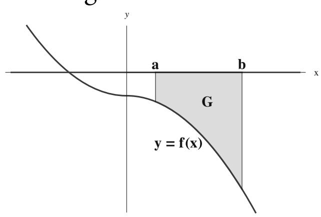


Figure 5: $\int_0^1 f(x) dx$ ***is the signed area of G***

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$, and $x_k^* \in [x_{k-1}, x_k], k = 1, 2, \ldots, n$. If $\|P\|$ is small, the Riemann sum

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

approximates

$$\int_a^b f(x) dx.$$

Consider the rectangle R_k that has the vertices $(x_{k-1}, 0), (x_k, 0), (x_{k-1}, f(x_k^*))$ and $(x_k, f(x_k^*))$, as in Figure 6.

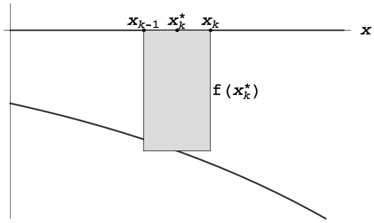


Figure 6

Since $f(x) \le 0$ for each $x \in [a, b]$, the term $f\left(x_x^*\right)\Delta x_k$ is (−1) × (the area of R_k). Thus, the Riemann sum

$$\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k$$

approximates (−1) × (area of G). We will refer to (−1) × (area of G) as **the signed area of *G*. Therefore, we will identify the integral of *f* on** [*a, b*] **with the signed area** of G:

$$\text{The signed area of } G = \int_a^b f(x)\,dx.$$

The area of *G* is

$$-\int_a^b f(x)\,dx.$$

Example 2 Let $f(x) = \sin(x)$. Figure 7 shows the region G between the graph of f and the interval $[\pi, 4\pi/3]$.

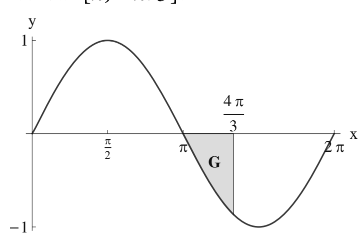


Figure 7: The signed area of G is −1/2

We have $\sin(x) \le 0$ if $\pi \le x \le 4\pi/3$. In Section 5.3 we will show that

$$\int_{\pi}^{4\pi/3} \sin(x)\,dx = -\frac{1}{2}.$$

Therefore, the signed area of G is $-1/2$, and the area of G is

$$-\int_{\pi}^{4\pi/3} \sin(x)\,dx = -\left(-\frac{1}{2}\right) = \frac{1}{2}.$$

Approximate the integral of sine on $[\pi, 4\pi/3]$ by midpoint sums that correspond to the partitioning of $[0, \pi]$ to 2^k subintervals of equal length, where $k = 2, \ldots, 6$.

Solution
We have

$$m_n = \sum_{k=1}^{n} \sin(c_k)\,\Delta x.$$

where

$$\Delta x = \frac{\frac{4\pi}{3} - \pi}{n} = \frac{\pi}{3n} \text{ and } c_k = \left(k - \frac{1}{2}\right)\Delta x,\ k = 1, 2, \ldots.n.$$

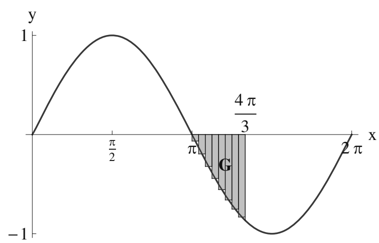


Figure 8

Table 1 displays m_n for $n = 4, 8, 16, 32$ and 64. The numbers in Table 1 are consistent with the fact that

$$\lim_{n\to\infty} m_n = \int_{\pi}^{4\pi/3} \sin(x)\,dx = -\frac{1}{2}.$$

□

n	m_n
4	−0.501431
8	−0.500357
16	−0.500089
32	−0.500022
64	−0.500006

Table 1

With reference to Figure 9, if a function f is positive-valued on $[a, b]$ we should have

$$(\text{area of } G_1) + (\text{area of } G_2) = \text{area of } G_1 \cup G_2.$$

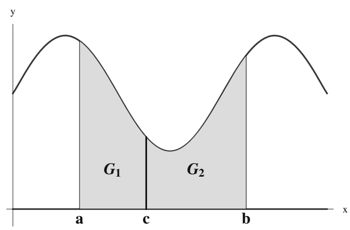


Figure 9: The integral is additive with respect to intervals

Thus, we expect that

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

This is indeed the case, irrespective of the sign of the function. We will refer to this property of the integral as "**the additivity of the integral with respect to intervals**".

Theorem 2 (The Additivity of the Integral with respect to Intervals) Assume that f is continuous on $[a, b]$ and $a < c < b$. Then

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

We leave the rigorous proof of Theorem 2 to a course in advanced calculus.

Let's assume that f is continuous on the interval $[a, b]$ and that the sign of f changes at a finite number of points in (a, b). In order to be specific, let's assume that $f(c) = 0$, $f(x) > 0$ on (a, c) and $f(x) < 0$ on

(c, b), as in Figure 10. With reference to Figure 10, the region G between the graph of f and the interval $[a, b]$ is the union of G_+ and G_-.

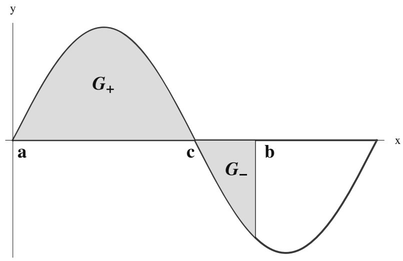


Figure 10: $\int_a^b f(x)\,dx$ ***= area of*** G_+ ***– area of*** G_-

By the additivity of the integral with respect to intervals,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Thus,

$$(\text{area of } G_+) + (\text{signed area of } G_-) = \int_a^b f(x)\,dx.$$

We will identify the signed area of the region $G = G_+ \cup G_-$ with the integral of f on $[a, b]$. The area of G is

$$\int_a^c f(x)\,dx - \int_c^b f(x)\,dx.$$

More generally, if a function f is continuous on an interval $[a, b]$, **we will identify the signed area of the region G between the graph of f and $[a, b]$ with the integral of f on $[a, b]$.** If we wish to compute the area of G, we must determine the subintervals of $[a, b]$ on which f has constant sign, and calculate the integral of f on each subinterval. The integral must be multiplied by -1 if the sign of f is negative on the relevant subinterval.

Example 3 Let $f(x) = \sin(x)$.

a) Sketch the region G between the graph of f and the interval $[0, 4\pi/3]$.

b) In Section 5.3 we will show that

$$\int_0^{\pi} \sin(x) = 2 \text{ and} \int_{\pi}^{4\pi/3} \sin(x)\,dx = -\frac{1}{2}$$

Determine the signed area and the area of G.

c) Approximate

$$\int_{\pi}^{4\pi/3} \sin(x)\,dx$$

by midpoint sums corresponding to the partitioning of the interval $[0, 4\pi/3]$ into 2^k subintervals of equal length, where $k = 3, \ldots, 7$.

Solution

a) Figure 11 shows the region G.

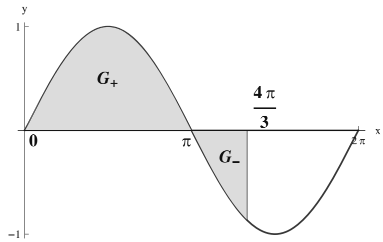


Figure 11

b) With reference to Figure 11,

$$\text{the area of } G_+ = \int_0^{\pi} \sin(x) = 2,$$

and

$$\text{the signed area of } G_- = \int_0^{4\pi/3} \sin(x)\,dx = -\frac{1}{2}.$$

Since $\sin(x) < 0$ if $\pi < x < 4\pi/3$, the area of G_- is 1/2.
The signed area of G is

$$\int_{\pi}^{4\pi/3} \sin(x)\,dx = \int_0^{\pi} \sin(x)\,dx + \int_{\pi}^{4\pi/3} \sin(x)\,dx = 2 + \left(-\frac{1}{2}\right) = \frac{3}{2},$$

and the area of G is

$$\text{area of } G_+ + \text{area of } G_- = \int_0^{\pi} \sin(x)\,dx - \int_0^{4\pi/3} \sin(x)\,dx = 2 - \left(-\frac{1}{2}\right) = \frac{5}{2}.$$

c) The midpoint sum corresponding to the partitioning of the interval $[0, 4\pi/3]$ to n subintervals of equal length is

$$m_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} \sin(c_k)\,\Delta x,$$

where

$$\Delta x = \frac{\frac{4\pi}{3}}{n} = \frac{4\pi}{3n} \text{ and } c_k = \left(k - \frac{1}{2}\right)\Delta x.$$

Table 2 displays m_n and

$$\left| m_n - \int_0^{4\pi/3} \sin(x)\,dx \right|$$

for $n = 2^k$, $k = 3, \ldots, 7$. The numbers in Table 2 support the expectation that

$$\lim_{n\to\infty} m_n = \int_0^{4\pi/3} \sin(x)\,dx.$$

□

n	Δx	m_n	$\lvert m_n - 1.5 \rvert$
8	.523 599	1.51727	1.7×10^{-2}
16	.261 799	1.50429	4.3×10^{-3}
32	.130 9	1.50107	1.1×10^{-3}
64	$6.544\,98 \times 10^{-2}$	1.50027	2.7×10^{-4}
128	$3.272\,49 \times 10^{-2}$	1.50007	6.7×10^{-5}

Table 2

Remark 1 In the notation

$$\int_c^b f(x)\,dx,$$

the variable x is a **dummy variable,** in the sense that the letter x can be replaced by any other letter. Thus, the expressions

$$\int_a^b f(x)\,dx,\ \int_a^b f(t)\,dt,\ \int_c^b f(\tau)\,d\tau$$

all have the same meaning: The integral of the function f on the interval $[a, b]$. For example,

$$\int_0^\pi \sin(x)\,dx = \int_0^\pi \sin(t)\,dt = \int_0^\pi \sin(u)\,du = 2.$$

This is parallel to the fact that the summation index for a Riemann sum is a dummy index, and we can use any letter to denote the independent variable of the function: The expressions

$$\sum_{k=1}^{n} f\left(x_k^*\right)\Delta x_k, \sum_{l=1}^{n} f\left(x_l^*\right)\Delta x_l, \sum_{j=1}^{n} f\left(t_j^*\right)\Delta x_j$$

have the same meaning. ◊

Remark 2 Your computational utility should be able to provide you with an accurate approximation to an integral. The underlying approximation schemes are referred to as **numerical integration schemes,** or **numerical integration rules.** We will see some of these rules in Section 6.5. A computer algebra system such as Maple or Mathematica is able to provide you with the exact value of many integrals. Soon, you will be able to compute the exact values of many integrals yourselves. ◊

The Integrals of Piecewise Continuous Functions

Theorem 1 states that a function which is continuous on a closed and bounded interval is (Riemann) integrable on that interval. It will be useful to expand the scope of the integral to a wider class of functions.

Assume that f is continuous on the interval (a, b) and

$$\lim_{x\to a+} f(x) \text{ and } \lim_{x\to b-} f(x)$$

exist. If we set

$$g(x) = \begin{cases} f(x) & \text{if} \quad x \in (a, b), \\ \lim_{x \to a+} f(x) & \text{if} \quad x = a, \\ \lim_{x \to b-} f(x) & \text{if} \quad x = b, \end{cases}$$

then g is continuous on $[a, b]$. We define the integral of f on $[a, b]$ to be the same as the integral of g on $[a, b]$:

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

This amounts to the fact that

$$\int_a^b f(x)\,dx$$

is approximated by Riemann sums of the form

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k,$$

where $f(x_0)$ should be interpreted as $\lim_{x \to a+} f(x)$ and $f(b)$ should be interpreted as $\lim_{x \to b-} f(x)$.

Example 4 Let

$$f(x) = \frac{\sin(x)}{x}$$

if $x \neq 0$.

a) Discuss the definition of $\int_0^\pi f(x)\,dx$.

b) Consider the approximate value of

$$\int_0^\pi \frac{\sin(x)}{x}\,dx$$

that you obtain from your computational utility to be the exact value of the integral. Approximate

$$\int_0^{\pi} \frac{\sin(x)}{x} dx$$

by midpoint sums corresponding to the partitioning of $[a, b]$ into 10, 20, 40 and 80 subintervals of equal length. Do the numbers support the fact that the integral can be approximated with desired accuracy by Riemann sums, provided that the norm of the partition is small enough?

Solution

a) Since sin (x) and x define continuous functions on the number line, the quotient f is continuous on the entire number line, with the exception $x = 0$. We have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

If we set

$$g(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \neq 0, \\ 1 & \text{if} \quad x = 0, \end{cases}$$

then g is continuous on $[0, \pi]$. We set

$$\int_0^{\pi} \frac{\sin(x)}{x} dx = \int_0^{\pi} g(x) dx.$$

The integral corresponds to the area of the region between the graph of f and the interval $[0, \pi]$, as illustrated in Figure 12.

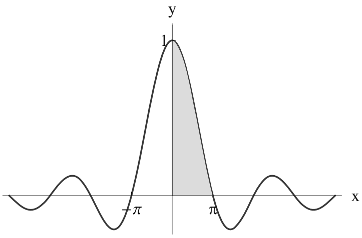


Figure 12

b) We have

$$m_n = \sum_{k=1}^{n} \frac{\sin(c_k)}{c_k} \Delta x,$$

where

$$\Delta x = \frac{\pi}{n} \text{and } c_k = \left(k - \frac{1}{2}\right)\Delta x.$$

Table 3 displays the relevant data. We have

$$\int_0^{\pi} \frac{\sin(x)}{x} dx \cong 1.851\,94,$$

rounded to 6 significant digits. The numbers in Table 3 support the fact that the integral can be approximated with desired accuracy by Riemann sums, provided that the norm of the partition is small enough. □

n	m_n	$\left\lvert m_n - \int_0^{\pi} f(x)\,dx \right\rvert$
10	1.85325	1.3×10^{-3}
20	1.85226	3.3×10^{-4}
40	1.85202	8.2×10^{-5}
80	1.85196	2.0×10^{-5}

Table 3

We will say that *f* is piecewise continuous on the interval [*a, b*] if *f* has at most finitely many removable or jump discontinuities in [*a, b*]. Thus, *f* has (finite) one-sided limits at its discontinuities. In such a case **we will define the integral of *f* on [*a, b*] as the sum of its integrals over the subintervals of [*a, b*] that are separated from each other by the points of discontinuity of *f*.**

Example 5 Let

$$f(x) = \begin{cases} \sin(x) & \text{if} \quad 0 \le x < \pi/2, \\ \cos(x) & \text{if} \quad \pi/2 < x < 3\pi/2. \end{cases}$$

Figure 13 shows the graph of f and the region between the graph of f and the interval $[0, 3\pi/2]$.

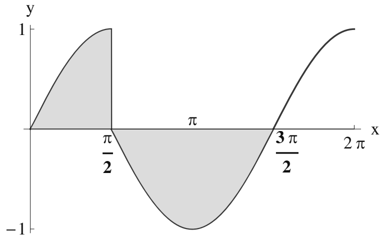


Figure 13

The function f is piecewise continuous on the interval $[0, 3\pi/2]$. Indeed, the only point of discontinuity of f in $[0, 3\pi/2]$ is $\pi/2$. We have

$$\lim_{x \to \pi/2-} f(x) = \lim_{x \to \pi/2} \sin(x) = 1,$$

and

$$\lim_{x \to \pi/2+} f(x) = \lim_{x \to \pi/2} \cos(x) = 0.$$

Therefore,

$$\int_0^{3\pi/2} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{3\pi/2} f(x)\,dx = \int_0^{\pi/2} \sin(x)\,dx + \int_{\pi/2}^{3\pi/2} \cos(x)\,dx.$$

It can be shown that

$$\int_0^{\pi/2} \sin(x)\,dx = 1 \text{ and } \int_{\pi/2}^{3\pi/2} \cos(x)\,dx = -2,$$

once we have developed the necessary tools in Section 5.3. Therefore,

$$\int_0^{3\pi/2} f(x)\,dx = 1 - 2 = -1.$$

Thus, the signed area of the region between the graph of f and the interval $[0, 2\pi]$ is -1. □

The Precise Definition of the Integral

We quantify the expressions "with desired accuracy" and "sufficiently small" that appear in the informal definition of the integral (Definition 2):

Definition 3 We say that a function f **is Riemann integrable on the interval** $[a, b]$ and that **the Riemann integral of** f **on** $[a, b]$ is

$$\int_a^b f(x)\,dx$$

if, given any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k - \int_a^b f(x)\,dx \right| < \varepsilon,$$

where $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a, b]$, $x_k^* \in \left[x_{k-1}, x_k\right]$, $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, \ldots, n$, and

$$\|P\| = \max_k \Delta x_k < \delta.$$

You may think of $\varepsilon > 0$ as an arbitrary "error tolerance" that is as small as desired. The positive δ that is referred to in the definition depends on ε, and must be sufficiently small so that the absolute value of the error in the approximation of the integral by *any* Riemann sum

$$\sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

is smaller than ε, provided that $\|P\| < \delta$. We should emphasize that there is complete freedom in the choice of the partition P and the choice of the intermediate points x_k^*, as long as $\|P\| < \delta$.

Remark 3 There are functions that are not Riemann integrable. For example, set

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational,} \\ 0 & \text{if} \quad x \text{ is irrational.} \end{cases}$$

We claim that f is not Riemann integrable on [0, 1]:

Let's set

$$x_k = \frac{k}{n}, k = 0, 1, 2, \cdots, n,$$

so that $\Delta x_k = 1/n$, $k = 1, 2, \ldots, n$, and

$$P_n = \{x_0, x_1, x_2, \cdots, x_n\} = \left\{1, \frac{1}{n}, \frac{2}{n}, \cdots, 1\right\}.$$

If each x_k^* is rational,

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k = \sum_{k=1}^{n} (1)\left(\frac{1}{n}\right) = n\left(\frac{1}{n}\right) = 1.$$

If each x_k^* is irrational,

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k = \sum_{k=1}^{n} (0) \Delta x_k = 0.$$

It can be shown that there are rational and irrational numbers in any interval, however small it may be. Since $\|P_n\| = 1/n$, and $\lim_{n\to\infty} 1/n = 0$, we can find partitions of arbitrarily small norm and corresponding Riemann sums that are 1 or 0. Therefore, we cannot assert that there is a definite number that is approximated by any Riemann sum with desired accuracy, provided that the norm of the relevant partition is small enough. This rules out the existence of the integral of the function. ◊

CHAPTER 3

Introduction to the Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (Part 1)

The first part of the Fundamental Theorem of Calculus states that **the integral of the derivative of a function on an interval is equal to the difference between the values of the function at the endpoints of the interval:**

Theorem 1 (THE FUNDAMENTAL THEOREM OF CALCULUS (Part 1)) **Assume that F' is continuous on** $[a, b]$ **Then**

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

$F'(a)$ and $F'(b)$ can be interpreted as the one sided derivatives $F'_+(a)$ and $F'_-(b)$, respectively.

The Proof of Theorem 1

Let $P = \{x_0, x_1, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}, x_n\}$ be a partition of $[a, b]$, so that $x_0 = a$ and $x_n = b$. We can express the change in the value of F over the interval $[a, b]$ as the sum of the changes in the value of F over the subintervals determined by P:

$$\begin{aligned}F(b)-F(a)&=F(x_n)-F(x_0)\\&=\left[F(x_n)-F(x_{n-1})\right]+\left[F(x_{n-1})-F(x_{n-2})\right]+\cdots\\&\qquad+\left[F(x_2)-F(x_1)\right]+\left[F(x_1)-F(x_0)\right]\\&=\sum_{k=1}^{n}\left[F(x_k)-F(x_{k-1})\right].\end{aligned}$$

By the Mean Value Theorem (Theorem 3 of Section 3.2), there exists $x_k^* \in (x_{k-1}, x_k)$ such that

$$F(x_k)-F(x_{k-1})=F'(x_k^*)(x_k-x_{k-1})=F'(x_k^*)\Delta x_k.$$

Therefore,

$$F(b)-F(a)=\sum_{k=1}^{n}F'(x_k^*)\Delta x_k.$$

We have

$$\sum_{k=1}^{n}F'(x_k^*)\Delta x_k \cong \int_a^b F'(x)\,dx$$

if $\|P\|$ = $\max_k \Delta x_k$ is small, and the approximation is as accurate as desired if $\|P\|$ is small enough. Therefore,

$$F(b)-F(a)\cong\int_a^b F'(x)dx,$$

and

$$\left|\left(F(b)-F(a)\right)-\int_a^b F'(x)\,dx\right|$$

is as small as desired. This means that the numbers

$$F(b)-F(a) \text{ and } \int_a^b F'(x)\,dx$$

are equal. ■

We may refer to the first part of the Fundamental Theorem of Calculus simply as "the Fundamental Theorem of Calculus" or "the Fundamental Theorem", until we introduce the second part of the Fundamental Theorem and a distinction is necessary.

Example 1 Let

$$F(x) = \frac{2}{3}x^{3/2}.$$

By the power rule,

$$F'(x) = \frac{d}{dx}\left(\frac{2}{3}x^{3/2}\right) = \frac{2}{3}\frac{d}{dx}x^{3/2} = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) = \sqrt{x}$$

if $x \geq 0$ (we have to interpret $F'(0)$ as $F'_+(0)$). Thus, F' is continuous on [0, 1], so that the Fundamental Theorem of Calculus is applicable on [0, 1]. Therefore,

$$\int_0^1 \sqrt{x}dx = \int_0^1 F'(x)dx = F(1) - F(0) = \frac{2}{3}.$$

Thus, the area of the region between the graph of $y = \sqrt{x}$ and the interval [0, 1] is 2/3. The region is illustrated in Figure 1. □

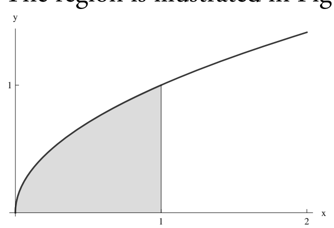


Figure 1: $\int_0^1 \sqrt{x} = \frac{2}{3}$

Example 2 Evaluate

$$\int_0^{\sqrt{\pi/4}} \frac{d}{dx}\cos\left(x^2\right)dx$$

Solution

If we set $F(x) = \cos(x^2)$, we have

$$\int_0^{\sqrt{\pi/4}} \frac{d}{dx}\cos\left(x^2\right)dx = \int_0^{\sqrt{\pi/4}} \frac{d}{dx}F(x)\,dx = F\left(\sqrt{\frac{\pi}{4}}\right) - F(0)$$

$$= \cos\left(\frac{\pi}{4}\right) - \cos(0) = \frac{\sqrt{2}}{2} - 1,$$

by Theorem 1.

Note that

$$f(x) = \frac{d}{dx}\cos\left(x^2\right) = -2x\sin\left(x^2\right),$$

and we have $f(x) \le 0$ if $0 \le x \le \sqrt{\pi/4}$. Thus, the area of the region G between the graph of f and the interval $\left[0, \sqrt{\pi/4}\right]$ is

$$-\int_0^{\sqrt{\pi/4}} f(x)\,dx = -\int_0^{\sqrt{\pi/4}} \frac{d}{dx}\cos\left(x^2\right)dx = 1 - \frac{\sqrt{2}}{2}.$$

Figure 2 shows the region G. □

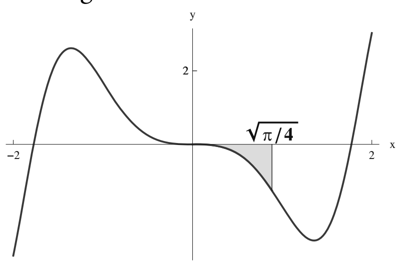


Figure 2

We were able to compute the integrals in the above examples by expressing the integrand as the derivative of a familiar function. We will use this procedure to compute many integrals:

Corollary (Corollary to the Fundamental Theorem of Calculus) Assume that f is continuous on $[a, b]$ and that $F'(x) = f(x)$ for each $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

By the Fundamental Theorem of Calculus (Part 1),

$$\int_a^b f(x)\,dx = \int_a^b F'(x)\,dx = F(b) - F(a)$$

■

As in Theorem 1, $F'(a)$ and $F'(b)$ can be interpreted as the one sided derivatives $F'_+(a)$ and $F'_-(b)$, respectively.

We may refer to the corollary to the Fundamental Theorem of Calculus simply as "the Fundamental Theorem of Calculus".

Definition 1 A function F is an **antiderivative** of f on an interval J if $\boldsymbol{F'(x) = f(x)}$ for each x in J.

The derivative should be interpreted as the appropriate one-sided derivative at an endpoint of the relevant interval.

We will denote $F(b) - F(a)$ as

$$F(x)\Big|_a^b.$$

Thus, we can express the Corollary to the Fundamental Theorem of Calculus as follows:

$$\boldsymbol{\int_a^b f(x)\,dx = F(x)\Big|_a^b}$$

if F is an antiderivative of f on $[a, b]$.

Example 3 Evaluate

$$\int_4^9 \sqrt{x}\,dx.$$

Solution

With reference to Example 1, if

$$f(x) = \sqrt{x} \text{ and } F(x) = \frac{2}{3}x^{3/2},$$

then F is an antiderivative of f on the interval $[0, +\infty)$, since

$$F'(x) = f(x)$$

for each $x \in (0, +\infty)$, and $F'_+(0) = f(0)$.

Therefore,

$$\int_4^9 \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big|_4^9 = \frac{2}{3}\left(9^{3/2} - 4^{3/2}\right) = \frac{2}{3}(27-8) = \frac{38}{8}.$$

□

We have been referring to "an antiderivative of a function". Indeed, a function has infinitely many antiderivatives. On the other hand, any two antiderivatives of the same function can differ at most by an additive constant:

Proposition 1 Let *F* be an antiderivative of *f* on the interval *J*.
a) If *C* is a constant, then *F* + *C* is also an antiderivative of *f* on *J*.
b) If *G* is any antiderivative of *f* on the interval *J*, there exists a constant *C* such that *G*(*x*) = *F* (*x*) + *C* for each *x* in *J*.

Proof

a) Since F is an antiderivative of f on J, we have

$$\frac{d}{dx}F(x) = f(x) \text{ for each } x \in J.$$

If C is an arbitrary constant,

$$\frac{d}{dx}\left(F(x)+C\right) = \frac{d}{dx}F(x) + \frac{d}{dx}(C) = f(x) + 0 = f(x)$$

for each x in J. Therefore, $F + C$ is also an antiderivative of f on the interval J.

b) Since F and G are antiderivatives of f on the interval J, we have

$$\frac{d}{dx}F(x) = f(x) \text{ and } \frac{d}{dx}G(x) = f(x)$$

for each $x \in J$. Therefore, there exists a constant C such that $G(x) = F(x) + C$ for all x in J (Corollary to Theorem 5 of Section 3.2). ■

By Proposition 1, if F is an antiderivative of f, we can express any antiderivative of f as $F + C$, where C is a constant. We will use the notation

$$\int f(x)\,dx$$

to denote *any* antiderivative of f and refer to

$$\int f(x)\,dx$$

as **the indefinite integral of** f. Thus,

$$\int f(x)\,dx = F(x) + C.$$

Example 4 If

$$F(x) = \frac{1}{3}x^3$$

and $f(x) = x^2$, then F is an antiderivative of f (on the entire number line), since

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3}\left(3x^2\right) = x^2$$

for each $x \in R$. Therefore, we can express the indefinite integral of f as

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

where C is an arbitrary constant. □

Remark 1 (Caution) We may refer to an integral

$$\int_a^b f(x)\,dx$$

as a **definite integral,** if we feel the need to make a distinction between an integral and an indefinite integral. In spite of the similarities between the terminology and the notation, **the indefinite integral of** f **and the**

integral of f on an interval $[a, b]$ **are distinct entities.** The (definite) integral

$$\int_a^b f(x)\,dx$$

is a number that can be approximated with arbitrary accuracy by Riemann sums, whereas, the indefinite integral

$$\int f(x)\,dx$$

represents any function whose derivative is equal to the function f. In either case, we will refer to f as **the integrand.** The Fundamental Theorem establishes a link between a definite integral and an indefinite integral:

$$\int_a^b f(x)\,dx = \int f(x)\Big|_{x=a}^{x=b}.$$

◊

Example 5 Let C denote an arbitrary constant. Show that the statements

$$\int 2\sin(x)\cos(x)\,dx = \sin^2(x) + C$$

and

$$\int 2\sin(x)\cos(x)\,dx = -\cos^2(x) + C$$

are both correct.

Solution
We have

$$\frac{d}{dx}\sin^2(x) = 2\sin(x)\cos(x)$$

and

$$\frac{d}{dx}\left(-\cos^2(x)\right) = -2\cos(x)\left(-\sin(x)\right) = 2\sin(x)\cos(x)$$

for each $x \in \mathbb{R}$. Therefore, both $\sin^2(x)$ and $-\cos^2(x)$ are antiderivatives for $2\sin(x)\cos(x)$. Therefore, we can express the indefinite integral of $2\sin(x)\cos(x)$ as

$$\int 2\sin(x)\cos(x)\,dx = \sin^2(x) + C$$

or

$$\int 2\sin(x)\cos(x)\,dx = -\cos^2(x) + C,$$

where C denotes an arbitrary constant.

Since $\sin^2(x)$ and $-\cos^2(x)$ are antiderivatives of the same function, they must differ by a constant. Indeed,

$$\sin^2(x) - \left(-\cos^2(x)\right) = \sin^2(x) + \cos^2(x) = 1$$

for all $x \in \mathbb{R}$. □

Remark 2 We should be able to use any antiderivative of the integrand in order to evaluate an integral. Indeed, if

$$\frac{d}{dx}F(x) = f(x) \text{ and } \frac{d}{dx}G(x) = f(x)$$

for each x in some interval J, there exists a constant C such that $G(x) - F(x) = C$ for each $x \in J$. Therefore,

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a),$$

and

$$\int_a^b f(x)\,dx = G(x)\Big|_{x=a}^{x=b} = G(b) - G(a) = \left(F(b) + C\right) - \left(F(a) + C\right) = F(b) - F(a).$$

Therefore, we do not have to include an arbitrary constant in the expression for an indefinite integral when we use the indefinite integral to evaluate a definite integral. ◊

Example 6 With reference to Example 5,

$$\int_{\pi/4}^{\pi/2} 2\sin(x)\cos(x)\,dx = \sin^2(x)\Big|_{\pi/2}^{\pi/2} = 1 - \left(\frac{1}{\sqrt{2}}\right)^2 = 1 - \frac{1}{2} = \frac{1}{2}.$$

We also have

$$\int_{\pi/4}^{\pi/2} 2\sin(x)\cos(x)\,dx = -\cos^2(x)\Big|_{\pi/2}^{\pi/2} = (0) + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}.$$

□

The Indefinite Integrals of Basic Functions

We will refer to the determination of the antiderivatives of functions as **antidifferentiation.** Traditionally, the term "integration" is also used instead of the term "antidifferentiation", even though we should make a distinction between integrals and antiderivatives. The particular context in which the terms are used should clarify the intended meaning. Antidifferentiation is not as straightforward as differentiation. Computer algebra systems are very helpful in finding the indefinite integrals of many functions. On the other hand, it is convenient to have the indefinite integrals of frequently encountered functions at your fingertips. Let's begin with a short list of indefinite integrals. You will learn about some rules of antidifferentiation in the rest of this chapter and in the next chapter. These rules will enable you to expand the scope of this short list considerably. The letter C denotes an arbitrary constant.

A Short List of Antiderivatives

1. $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C,\ r \neq -1$ (if x^r is defined)
2. $\int \frac{1}{x}\,dx = \ln(|x|) + C$ (on any interval that does not contain 0)
3. $\int \sin(\omega x)\,dx = \frac{1}{\omega}\cos(\omega x) + C$ (ω is a nonzero constant)

4. $\int \cos(\omega x)\,dx = \frac{1}{\omega}\sin(x) + C$ (ω is a nonzero constant)
5. $\int e^x\,dx = e^x + C$
6. $\int a^x\,dx = \frac{1}{\ln(a)}a^x + C$, where $a > 0$
7. $\int \sinh(x)\,dx = \cosh(x) + C$
8. $\int \cosh(x)\,dx = \sinh(x) + C$
9. $\int \frac{1}{x^2+1}\,dx = \arctan(x) + C$

By the definition of the indefinite integral, each formula is confirmed by differentiation.

1. Let J be an interval that is contained in the natural domain of x^r. By the power rule,

$$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = \frac{1}{r+1}\frac{d}{dx}\left(x^{r+1}\right) = \frac{1}{r+1}(r+1)x^r = x^r$$

for each x in J (the derivative may have to be interpreted as a one-sided derivative at 0). Therefore,

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$$

on the interval J. We will refer to the above antidifferentiation rule as **the reverse power rule** since it is a consequence of the power rule for differentiation.

2. If $x > 0$,

$$\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(x) = \frac{1}{x}.$$

If $x < 0$,

$$\frac{d}{dx}\ln(|x|) = \frac{d}{dx}\ln(-x) = \left(\frac{d}{du}\ln(u)\Big|_{u=-x}\right)\left(\frac{d}{dx}(-x)\right) = \left(\frac{1}{-x}\right)(-1) = \frac{1}{x},$$

with the help of the chain rule.
Therefore,

$$\int \frac{1}{x}\,dx = \ln\left(|x|\right) + C$$

on any interval that does not contain 0.
Figure 3 shows the graph of $y = \ln\left(|x|\right)$. Note that $\ln\left(|x|\right)$ defines an even function, so that the graph of the function is symmetric with respect to the vertical axis. Also note that

$$\lim_{x\to 0-} \ln\left(|x|\right) = \lim_{x\to 0+} \ln\left(|x|\right) = -\infty.$$

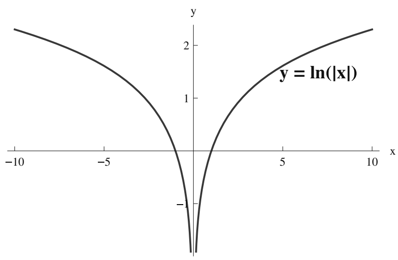


Figure 3: y = ln (| x |)

Formulas 3 - 9 are equivalent to the following differentiation formulas, respectively:

$$\frac{d}{dx}\left(-\frac{1}{\omega}\cos(\omega x)\right) = \sin(\omega x),$$

$$\frac{d}{dx}\left(\frac{1}{\omega}\sin(\omega x)\right) = \cos(\omega x),$$

$$\frac{d}{dx}e^x = e^x,$$

$$\frac{d}{dx}\left(\frac{1}{\ln(a)}a^x\right) = \frac{1}{\ln(a)}\frac{d}{dx}a^x = \frac{1}{\ln(a)}\left(\ln(a)a^x\right) = a^x,$$

$$\frac{d}{dx}\cosh(x) = \sinh(x),$$

$$\frac{d}{dx}\sinh(x) = \cosh(x),$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1},$$

Example 7

a) Determine

$$\int x^{2/3}dx$$

by the reverse power rule. Confirm the result by differentiation.

b) Compute

$$\int_{-8}^{27} x^{2/3}dx.$$

Interpret the integral as signed area.

Solution

a) By the reverse power rule,

$$\int x^{2/3}dx = \frac{x^{2/3+1}}{2/3+1} = \frac{x^{5/3}}{5/3} = \frac{3}{5}x^{5/3} + C,$$

where C is an arbitrary constant.
We have

$$\frac{d}{dx}\left(\frac{3}{5}x^{5/3} + C\right) = \frac{3}{5}\left(\frac{5}{3}x^{2/3}\right) = x^{2/3}$$

for each $x \in \mathbb{R}$. Therefore, the statement

$$\int x^{2/3}dx = \frac{3}{5}x^{5/3} + C$$

is valid on $\mathbb{R}$.

b) By the Fundamental Theorem of Calculus,

$$\int_{-8}^{27} x^{2/3}dx = \frac{3}{5}x^{5/3}\Bigg|_{-8}^{27} = \frac{3}{5}\left(27^{5/3} - (-8)^{5/3}\right) = \frac{3}{5}\left(3^5 + 2^5\right) = 165.$$

Note that f is continuous on $\mathbb{R}$, even though f is not differentiable at 0, so that there is no problem about the existence of an integral of f, or the application of the Fundamental Theorem. Since $x^{2/3} \geq 0$ for each x, the area of the region between the graph of $f(x) = x^{2/3}$ and the interval $[-8, 27]$ is 165. □

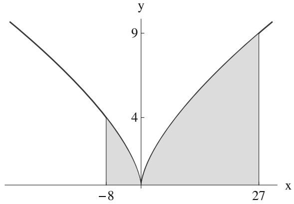


Figure 4: The region between the graph of $y = x^{2/3}$ and $[-8, 27]$

Example 8

a) Determine

$$\int \frac{1}{x^2}\,dx,$$

and the intervals on which the expression is valid.

b) Compute

$$\int_{-2}^{-1} \frac{1}{x^2}\,dx.$$

Interpret the integral as signed area.

Solution

a) By the reverse power rule,

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} = -\frac{1}{x} + C,$$

where C is an arbitrary constant. The expression

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$$

is valid on the interval $(-\infty, 0)$ and on the interval $(0, +\infty)$.

b) By the Corollary to the Fundamental Theorem of Calculus,

$$\int_{-2}^{-1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Bigg|_{-2}^{-1} = \left(-\frac{1}{(-1)}\right) - \left(-\frac{1}{(-2)}\right) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Since $1/x^2 > 0$, the area of the region between the graph of $f(x) = 1/x^2$ and the interval $[-2, -1]$ is 0.5. Figure 5 illustrates the region. □

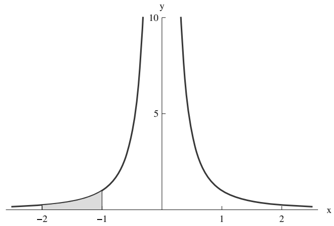


Figure 5

Example 9 Since $x^{-2} > 0$, the following claim cannot be valid:

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{-1}^{2} = \left(-\frac{1}{2}\right) - (1) = -\frac{3}{2}.$$

Why is the above line incorrect?

Solution

We have

$$\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{1}{x}$$

if and only if $x \neq 0$. But 0 is in the interval $[-1, 2]$, so that the Corollary to the Fundamental Theorem of Calculus (Corollary) cannot be applied as indicated above. □

Example 10 Evaluate

$$\int_{-4}^{-2} \frac{1}{x}\,dx.$$

Solution

Since

$$\int \frac{1}{x}\,dx = \ln(|x|) + C,$$

on any interval contained in $(-\infty, 0)$ or $(0, +\infty)$, and $[-4, -2]$ is contained in $(-\infty, 0)$, we can use the above indefinite integral to evaluate the given definite integral. By the Fundamental Theorem of Calculus,

$$\int_{-4}^{-2} \frac{1}{x} dx. = \ln(|x|)\Big|_{-4}^{-2} = \ln(|-2|) - \ln(|-4|) = \ln(2) - \ln(4) \cong -0.693147$$

Thus, the signed area of the region between the graph of the function defined by $1/x$ and the interval $[-4, -2]$ is ln (2) – ln(4). The region is illustrated in Figure 6. □

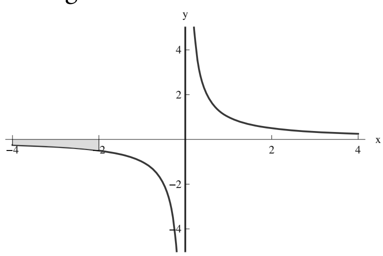


Figure 6

Remark 3 (Caution) We must be careful with the use of the antidifferentiation formula,

$$\int \frac{1}{x} dx = \ln(|x|) + C.$$

For example, we might be tempted to write,

$$\int_{-2}^{3} \frac{1}{x} dx = \ln(|x|)\Big|_{-2}^{3} = \ln(3) - \ln(2).$$

The above statement is not valid since it is not true that

$$\frac{d}{dx}\ln(|x|) = \frac{1}{x}$$

at 0, and $0 \in (-2, 3)$. ◊

Example 11 Evaluate

$$\int_0^{\ln(10)} e^x dx.$$

Solution
We have

$$\int e^x dx = e^x + C,$$

where C is an arbitrary constant. By the Fundamental Theorem of Calculus,

$$\int_0^{\ln(10)} e^x dx = e^x \Big|_{x=0}^{\ln(10)} = e^{\ln(10)} - e^0 = 10 - 1 = 9.$$

Thus, the area of the region between the graph of the natural exponential function and the interval [0, ln(10)] is 9. Figure 7 illustrates the region. □

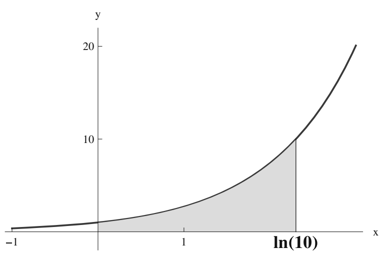


Figure 7

Example 12 Confirm the following claims that were made in Example 4 of Section 5.2:

$$\int_0^{\pi} \sin(x)\, dx = 2 \text{ and } \int_{\pi}^{4\pi/3} \sin(x)\, dx = -\frac{1}{2}.$$

Solution
We have

$$\int \sin(x)\, dx = -\cos(x) + C,$$

where C denotes an arbitrary constant, as usual. By the Fundamental Theorem of Calculus,

$$\int_0^{\pi} \sin(x)\,dx = -\cos(x)\Big|_0^{\pi} = -\cos(\pi) - (-\cos(0)) = 1 + 1 = 2,$$

and

$$\int_{\pi}^{4\pi/3} \sin(x)\,dx = -\cos(x)\Big|_{\pi}^{4\pi/3} = -\cos\left(\frac{4\pi}{3}\right) - (-\cos(\pi)) = -\left(-\frac{1}{2}\right) - 1 = -\frac{1}{2}.$$

□

The Fundamental Theorem of Calculus and One-Dimensional Motion

Let's interpret the Fundamental Theorem of Calculus within the context of one-dimensional motion. Assume that $f(t)$ is the **position** at time t of an object in one dimensional motion, and let $v(t)$ be its instantaneous **velocity** at time t, so that $v(t) = f'(t)$. Also assume that v is continuous on $[a, b]$. By the Fundamental Theorem of Calculus,

$$\int_a^b v(t)\,dt = \int_a^b f'(t)\,dt = f(b) - f(a).$$

We will refer to the change in the position of the object over the time time interval $[a, b]$ as **the displacement** of the object over that time interval. Thus, **the displacement of the object over the time interval $[a, b]$ is equal to the integral of the velocity function on $[a, b]$.**

Even though the above fact is a direct consequence of the Fundamental Theorem of Calculus, it is helpful to interpret the proof of the theorem within the context of one-dimensional motion. If $P = \{t_0, t_1, t_2, \ldots, t_{n-1}, t_n\}$ is a partition of $[a, b]$, so that $t_0 = a$ and $t_n = b$, we can express the displacement over $[a, b]$ as the sum of the displacements over the subintervals. Thus,

$$\begin{aligned}f(b)-f(a)&=f(t_n)-f(t_0)\\&=\left[f(t_n)-f(t_{n-1})\right]+\left[f(t_{n-1})-f(t_{n-2}))\right]+\cdots\\&\quad+\left[f(t_2)-f(t_1)\right]+\left[f(t_1)-f(t_0)\right]\\&=\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right].\end{aligned}$$

By the Mean Value Theorem, there exists $t_k^* \in (t_{k-1}, -t_k)$ such that

$$f(t_k)-f(t_{k-1})=f'(t_k^*)(t_k-t_{k-1})=\upsilon(t_k^*)\Delta t_k.$$

Therefore,

Displacement over $[a,b]=\sum_{k=1}^{n}\left[f(t_k)-f(t_{k-1})\right]=\sum_{k=1}^{n}\upsilon(t_k^*)\Delta t_k.$

Since

$$\sum_{k=1}^{n}\upsilon(t_k^*)\Delta t_k \cong \int_a^b \upsilon(t)\,dt$$

if $\|P\|$ is small, and the approximation is as accurate as desired provided that $\|P\|$ is small enough,

$$\left|\text{Displacement over } [a,b]-\int_a^b \upsilon(t)\,dt\right|$$

is arbitrarily small. This is the case if and only if

$$\text{Displacement over } [a,b]=\int_a^b \upsilon(t)\,dt.$$

In particular the units match. For example, if distance is measured in centimeters and time is measured in seconds, velocity is expressed in terms of centimeters per second. This is consistent with the fact that

$$\text{Displacement over } [a,b]=\int_a^b \upsilon(t)\,dt \cong \sum_{k=1}^{n}\upsilon(t_k^*)\Delta t_k.$$

Indeed, the unit of $\upsilon(t_k^*)\Delta t_k$ is

$$\frac{\text{centimeter}}{\text{second}}\times\text{second} = \text{centimeter}.$$

Graphically, the displacement of the object over the time interval [*a, b*] is **the signed area** of the region between the graph of the velocity function and the interval [*a, b*]. We must distinguish between the displacement of an object over a time interval and **the distance traveled** by the object over the same time interval. If $v(t) \le 0$ for each $t \in [a, b]$, the object is moving in the negative direction over the time interval [*a, b*]. Therefore, the distance traveled is

$$-\int_a^b v(t)\,dt.$$

More generally, if we wish to calculate the distance traveled by an object over the time interval [*a, b*], we need to determine the subintervals of [*a, b*] on which the velocity has constant sign. If the velocity is negative over a subinterval, the relevant integral must be multiplied by (–1). Graphically, the distance traveled over the time interval [*a, b*] is **the area** between the graph of the velocity function and the interval [*a, b*].

Example 13 With the above notation, assume that an object that is attached to a spring has velocity $v(t) = \cos(2t)$.

a) Sketch the graph of the velocity function on $[0, \pi]$.
b) Determine the displacement of the object over the time interval $[0, 3\pi/4]$.
c) Determine the distance traveled by the object over the time interval $[0, 3\pi/4]$.

Solution

a) Figure 8 shows the graph of the velocity function on $[0, \pi]$.

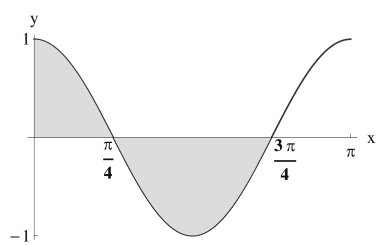


Figure 8

b) Since

$$\int \cos(\omega t)\,dt = \frac{1}{\omega}\sin(\omega t) + C,$$

for any $\omega \neq 0$, we have

$$\int \cos(2t)\,dt = \frac{1}{2}\sin(2t) + C.$$

Therefore, the displacement of the object over the time interval $[0, 3\pi/4]$ is

$$\int_0^{3\pi/4} v(t)\,dt = \int_0^{3\pi/4} \cos(2t)\,dt = \frac{1}{2}\sin(2t)\Big|_0^{3\pi/4}$$
$$= \frac{1}{2}\sin\left(\frac{3\pi}{2}\right) - \frac{1}{2}\sin(0) = -\frac{1}{2}$$

(centimeters).

c) We see that $v(t) > 0$ if $0 < t < \pi/4$ and $v(t) < 0$ if $\pi/4 < t < 3\pi/4$. Thus, the object is moving in the positive direction over the time interval $[0, \pi/4]$ and in the negative direction over the time interval $[\pi/4, 3\pi/4]$. We have

$$\int_0^{\pi/4} v(t)\,dt = \frac{1}{2}\sin(2t)\Big|_0^{\pi/4} = \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0) = \frac{1}{2},$$

and

$$\int_{\pi/4}^{3\pi/4} v(t)\,dt = \frac{1}{2}\sin(2t)\Big|_{\pi/4}^{3\pi/4} = \frac{1}{2}\sin\left(\frac{3\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2} - \frac{1}{2} = -1.$$

Therefore, total distance traveled is

$$\int_0^{\pi/4} v(t)\,dt - \int_{\pi/4}^{3\pi/4} v(t)\,dt = \frac{1}{2} - (-1) = \frac{1}{2}$$

(centimeters). Graphically, the distance traveled is the area of the region between the velocity function and the interval $[0, 3\pi/4]$. □

CHAPTER 4

The Antiderivative and the Fundamental Theorem of Calculus

The second part of the Fundamental Theorem of Calculus shows that every continuous function has an antiderivative, even though such an antiderivative may not be expressible in terms of familiar functions. The theorem leads to the definition of new special functions.

Some Properties of the Integral

In preparation for the second part of the Fundamental Theorem of Calculus, we will discuss some general facts about the integral that will be useful in other contexts as well.

Proposition 1 Assume that f and g are continuous on the interval [a, b] and $f(x) \leq g(x)$ for each $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Figure 1 illustrates the graphical meaning of Proposition 1 if $0 \leq f(x) < g(x)$ for each $x \in [a, b]$: The area of the region between the graph of f and the interval $[a, b]$ is less than the area of the region between the graph of g and $[a, b]$.

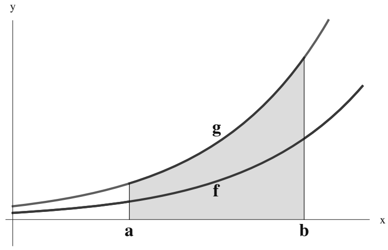


Figure 1

We will leave the rigorous proof of Proposition 1 to a course in advanced calculus. Let's provide a plausibility argument:

Let $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ be a partition of $[a, b]$ and $x_k^* \in [x_{k-1}, x_k], k = 1, 2, \ldots, n.$ We have

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k \leq \sum_{k=1}^{n} g(x_k^*) \Delta x_k,$$

since $f(x) \leq g(x)$ for each $x \in [a, b]$. Since

$$x \sum_{k=1}^{n} f(x_k^*) \Delta x_k \cong \int_a^b f(x)\, dx \text{ and } \sum_{k=1}^{n} g(x_k^*) \Delta x_k \cong \int_a^b g(x)\, dx,$$

if $||P|| = \max_k \Delta x_k$ is small, it is plausible that

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

■

Corollary 1 (The Triangle Inequality for Integrals) Assume that f is continuous on [a, b]. Then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof

It can be shown that $|f|$ is continuous on $[a, b]$ if f is continuous on $[a, b]$. We have

$$-|f(x)| \leq f(x) \leq |f(x)|$$

for each $x \in [a, b]$. By Proposition 1,

$$\int_a^b -|f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx.$$

By the constant multiple rule for integrals,

$$\int_a^b -|f(x)|\, dx = -\int_a^b |f(x)|\, dx.$$

Therefore,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

The above inequalities imply that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

■

We have dubbed the Corollary 1 as "**the triangle inequality for integrals**", since we can view the inequality

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

as a generalization of the triangle inequality for numbers. Indeed, if $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ is a partition of $[a, b]$ and $x_k^* \in [x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$, we have

$$\left|\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k\right| \le \sum_{k=1}^{n} |f(x_k^*)|\,\Delta x_k$$

by the triangle inequality for numbers. If $\|P\| = \max_k \Delta x_k$ is small,

$$\left|\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k\right| \cong \left|\int_a^b f(x)\,dx\right|$$

and

$$\sum_{k=1}^{n} |f(x_k^*)|\,\Delta x_k \cong \int_a^b |f(x)|\,dx.$$

Therefore, the inequality

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

is not surprising.

Definition 1 The mean value (or **the average value**) of a continuous function *f* on the interval [*a, b*] is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Thus, the mean value of *f* on [*a, b*] is the ratio of the integral of *f* on [*a, b*] and the length of the interval [*a, b*].

The terminology of Definition 1 is reasonable. Indeed, if

$$\Delta x = \frac{b-a}{n},\ x_k = a + k\Delta x,\ k = 1, 2, \cdots, n,$$

then

$$\sum_{k=1}^{n} f(x_k)\,\Delta x \cong \int_a^b f(x)\,dx$$

if Δx is small, i.e., n is large. Therefore,

$$\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\,\Delta x \cong \frac{1}{b-a}\int_a^b f(x)dx.$$

We have

$$\frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\,\Delta x = \frac{1}{b-a}\sum_{k=1}^{n} f(x_k)\left(\frac{b-a}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} f(x_k).$$

Therefore,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \cong \frac{1}{b-a}\int_a^b f(x)\,dx$$

if n is large. The quantity

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k)$$

is the mean of the values of the function at the points x_k, $k = 1, 2, \ldots, n$.

A Continuous function attains its mean value on an interval:
Theorem 1 (THE MEAN VALUE THEOREM FOR INTEGRALS) **Assume that f is continuous on $[a, b]$. There exists $c \in [a, b]$ such that**

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof

Let m and M be the minimum and the maximum value of f on $[a, b]$, respectively. Since

$$m \le f(x) \le M$$

for each $x \in [a, b]$, we have

$$\int_a^b m dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx,$$

by Proposition 1. Therefore,

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Thus,

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

By the Intermediate Value Theorem for continuous functions (Theorem 1 of Section 2.9), there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

■

Since

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \Rightarrow f(c)(b-a) = \int_a^b f(x)\,dx,$$

we can interpret the Mean Value Theorem for Integrals in the case of a positive-valued function f graphically: The area of the region between the graph of f and the interval $[a, b]$ is the same as the area of a rectangle that has as its base the interval $[a, b]$ and has height equal to the value of f at some c in $[a, b]$, as illustrated in Figure 2.

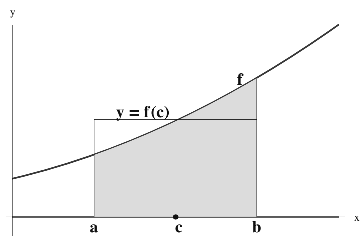


Figure 2

An integral is multiplied by (−1) if the upper and lower limits are interchanged:

Definition 2 Assume that $a < b$. We define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Remark 1 If F is an antiderivative of f, we have

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = -\big(F(b) - F(a)\big) = F(a) - F(b).$$

Therefore,

$$\int_b^a f(x)\,dx = F(x)\Big|_b^a,$$

just as

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b.$$

Thus, we need not pay attention to the positions of a and b on the number line relative to each other, when we make use of the Fundamental Theorem to evaluate the integral. ◊

Remark 2 By Definition 2, if ***v***(***t***) is the **velocity** at time *t* of an object in **one-dimensional motion,** and ***f*** is the corresponding **position function,** we have

$$\int_b^a v(t)\,dt = -\int_a^b v(t)\,dt = -\int_a^b f'(t)\,dt = -\left(f(b) - f(a)\right) = f(a) - f(b).$$

Thus, if we imagine that time flows backwards from *b* to *a,* the integral of the velocity function from *b* to *a* is still the change in the position function. ◊

Example 1 Determine

$$\int_{\pi/2}^0 \cos(x)\,dx.$$

Solution

By Definition 2,

$$\int_{\pi/2}^0 \cos(x)\,dx = -\int_0^{\pi/2} \cos(x)\,dx = -\left(\int \cos(x)\,dx \Big|_{x=0}^{x=\pi/2}\right)$$
$$= -\left(\sin(x)\Big|_0^{\pi/2}\right) = -\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right) = -1.$$

We can obtain the same result as follows:

$$\int_{\pi/2}^0 \cos(x)\,dx = \sin(x)\Big|_{\pi/2}^0 = \sin(0) - \sin\left(\frac{\pi}{2}\right) = -1.$$

□

We define an integral that has the same lower and upper limits to be 0:

Definition 3

$$\int_a^a f(x)\,dx = 0.$$

The following argument suggests that the above definition is reasonable:

Assume that *f* is continuous in some open interval that contains the point *a* and that $|f(x)| \le M$ for each *x* in that interval. If the positive integer *n* is large enough,

$$\left|\int_{a-1/n}^{a+1/n} f(x)\,dx\right| \le \int_{a-1/n}^{a+1/n} |f(x)|\,dx \le \int_{a-1/n}^{a+1/n} M dx = M\left(\frac{2}{n}\right),$$

with the help of the triangle inequality for integrals. Therefore,

$$\lim_{n\to\infty}\int_{a-1/n}^{a+1/n} f(x)\,dx = 0.$$

Thus, it is natural to set

$$\int_a^a f(x)\,dx = \lim_{n\to\infty}\int_{a-1/n}^{a+1/n} f(x)\,dx = 0.$$

■

The above definitions enable us to express the generalized version of **the additivity of the integral with respect to intervals:**

Theorem 2 If *f* is continuous on an interval that contains the points *a, b* and *c,* we have

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Proof

We know that the statement of Thorem 2 is valid if $a < b < c$. Assume that $a < c < b$. Then,

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Therefore,

$$\int_a^c f(x)dx = \int_a^b f(x)\,dx - \int_c^b f(x)\,dx = \int_a^b f(x)\,dx - \left(-\int_b^c f(x)\,dx\right)$$
$$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx,$$

as claimed.

Let's consider the case $a = b < c$. Then,

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^a f(x)\,dx + \int_a^c f(x)\,dx = 0 + \int_a^c f(x)dx = \int_a^c f(x)dx.$$

Other cases are handled in a similar fashion. ■

The Second Part of the Fundamental Theorem

We will define functions via integrals. Assume that f is continuous on an interval J that contains the point a. Let us set

$$F(x) = -\int_a^x f(t)\, dt,$$

for each $x \in J$. Note that the upper limit of the integral is the variable x, and we used the letter t to denote the "dummy" integration variable (we could have used any letter other than x). If $x > a$, then $F(x)$ is the signed area of the region between the graph of f and the interval $[a, x]$, as illustrated in Figure 3.

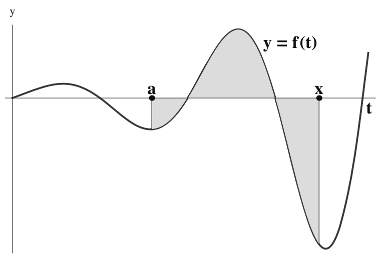


Figure 3: $F(x) = -\int_a^x f(t)\, dt,$

If $x < a$, we have

$$F(x) = -\int_x^a f(x)\, dx$$

so that $F(x)$ is (−1) times the signed area of the region between the graph of f and the interval $[a, x]$, as illustrated in Figure 4.

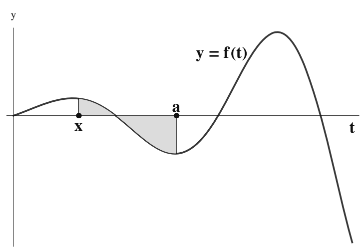


Figure 4: $F(x) = -\int_x^a f(x)\, dx$

Note that

$$F(a) = \int_a^a f(t)\,dt = 0.$$

Example 2 Set

$$F(x) = \int_2^x t^2\,dt.$$

a) Determine $F(x)$, $F(3)$ and $F(1)$.
b) Interpret the meaning of $F(x)$ graphically. Sketch the graph of F.
c) Determine $F'(x)$.

Solution

a) By the reverse power rule,

$$\int t^2\,dt = \frac{t^3}{3} + C,$$

where C is an arbitrary constant. Therefore,

$$F(x) = \int_2^x t^2\,dt = \left.\frac{t^3}{3}\right|_2^x = \frac{x^3}{3} - \frac{2^3}{3} = \frac{1}{3}x^3 - \frac{8}{3}.$$

In particular,

$$F(3) = \int_2^3 t^2\,dt = \frac{19}{3} \text{ and } F(1) = \int_2^1 t^2\,dt - \frac{7}{3}.$$

b) We have

$$F(2) = \int_2^2 t^2\,dt = 0.$$

If $x > 2$, then $F(x)$ is the area between the graph of $f(t) = t^2$ and the interval $[2, x]$, as illustrated in Figure 5.

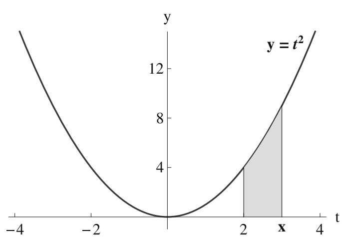


Figure 5: $F(x) = \int_2^x t^2\,dt$

If $x < 2$, then

$$F(x) = \int_x^2 t^2 \, dt$$

Therefore, $F(x)$ is (-1) times the area of the region between the graph of $f(t) = t^2$ and the interval $[x, 2]$, as illustrated in Figure 6.

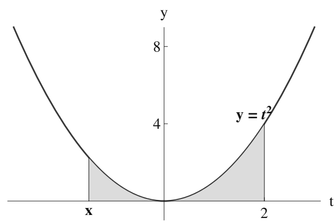


Figure 6: $F(x) = \int_x^2 t^2 \, dt$ if $x < 2$

Figure 7 shows the graph of F.

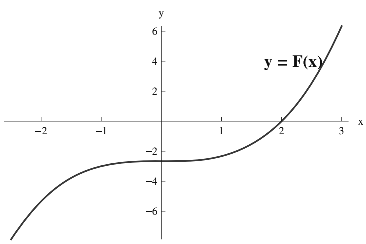


Figure 7: $y = F(x) = \int_2^x t^2 \, dt$

c) We have

$$F'(x) = \frac{d}{dx}\int_2^x t^2 \, dt = \frac{d}{dx}\left(\frac{1}{3}x^3 - \frac{8}{3}\right) = \frac{1}{3}\left(3x^2\right) = x^2.$$

Note that x^2 is the value of the integrand t^2 at $t = x$. □

Example 3 Set

$$F(x) = \int_\pi^x \sin(t) \, dt.$$

a) Determine $F(x)$, $F(3\pi/2)$ and $F(\pi/3)$.

b) Interpret the meaning of $F(x)$ graphically. Sketch the graph of F on the interval $[0, 3\pi]$.

c) Determine $F'(x)$.

Solution

a) We have

$$F(x) = \int_{\pi}^{x} \sin(t)\, dt = -\cos(t)\Big|_{\pi}^{x} = -\cos(x) + \cos(\pi) = -\cos(x) - 1.$$

In particular,

$$F(3\pi/2) = \int_{\pi}^{3\pi/2} \sin(t)\, dt = -1 \text{ and } F(\pi/3) = \int_{\pi}^{\pi/3} \sin(t)\, dt = -\frac{3}{2}.$$

b) We have

$$F(\pi) = \int_{\pi}^{\pi} \sin(t)\, dt = 0.$$

If $x > \pi$, $F(x)$ is the signed area of the region between the graph of sine and the interval $[\pi, x]$, as illustrated in Figure 8.

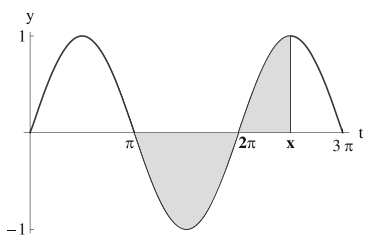


Figure 8: $F(x) = \int_{\pi}^{x} \sin(t)\, dt$

If $x < \pi$, we have

$$F(x) = \int_{\pi}^{x} \sin(t)\, dt = -\int_{x}^{\pi} \sin(t)\, dt.$$

Therefore, $F(x)$ is (-1) times the signed area of the region between the graph of sine and the interval $[x, \pi]$, as illustrated in Figure 9.

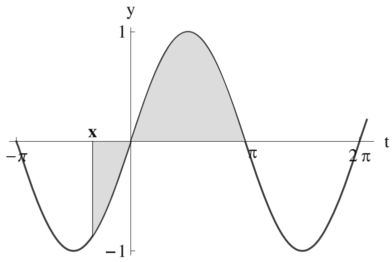


Figure 9: $F(x) = -\int_{\pi}^{\pi} \sin(t)\,dt$

Figure 10 shows the graph of $y = F(x) = -\cos(x) - 1$ on the interval $[-2\pi, 2\pi]$.

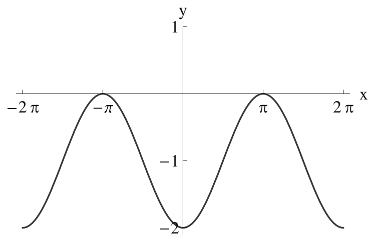


***Figure 10:* $y = F(x) = -\cos(x) - 1$**

c) We have

$$F'(x) = \frac{d}{dx}\int_{\pi}^{x} \sin(t)\,dt = \frac{d}{dx}\left(-\cos(x) - 1\right) = \sin(x).$$

Note that $\sin(x)$ is the value of the integrand $\sin(t)$ at $t = x$. □

In examples 2 and 3, it turned out that

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x).$$

That is a general fact:

Theorem 3 (The Fundamental Theorem of Calculus, Part 2) Assume that f is continuous on the interval J, and a is a given point in J. If

$$F(x) = \int_{a}^{x} f(t)\,dt,$$

then $F'(x) = f(x)$ for each $x \in J$.

The derivative should be interpreted as the appropriate one-sided derivative at an endpoint of *J*.

Remark 3 The second part of the Fundamental Theorem of Calculus asserts that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

for each $x \in J$ (provided that *f* is continuous on *J*). Therefore,

$$F(x) = \int_a^x f(t)\,dt$$

defines an antiderivative of *f* on *J*. ◊

A Plausibility Argument for Theorem 3:

We have

$$F(x+\Delta x) - F(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt.$$

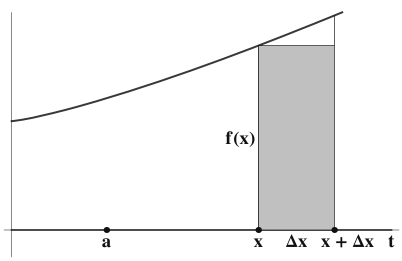


Figure 11: $\int_x^{x+\Delta x} f(t)\,dt \cong f(x)\,\Delta x$

With reference to Figure 11, if $\Delta x > 0$ and small, this quantity is approximately the area of the rectangle that has as its base the interval $[x, x + \Delta x]$ and has height $f(x)$. Therefore

$$F(x+\Delta x) - F(x) \cong f(x)\Delta x,$$

so that

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} \cong f(x)$$

if Δx is small. Thus, it is plausible that

$$F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x).$$

■

The Proof of Theorem 3

We will show that $F'(x) = f(x)$ at a point x in the interior of J. If x is an endpoint of J, the equality of the appropriate one-sided derivative of F and $f(x)$ is established in a similar manner.

Let $\Delta x > 0$. As in our plausibility argument,

$$F(x+\Delta x) - F(x) = \int_x^{x+\Delta x} f(t)\,dt.$$

Therefore,

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)\,dt.$$

Thus, the difference quotient is the mean value of f on the interval $[x, x + \Delta x]$. By the Mean Value Theorem for Integrals (Theorem 1), there exists a point $c(x, \Delta x)$ in the interval $[x, x + \Delta x]$ such that

$$\frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)\,dt = f\left(c\left(x, \Delta x\right)\right)$$

(we have used the notation "$c(x, \Delta x)$" in order to indicate that c depends on x and Δx).

Therefore,

$$F_+'(x) = \lim_{\Delta x \to 0+} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0+} f\left(c\left(x, \Delta x\right)\right).$$

Since $c(x, \Delta x)$ is between x and $x + \Delta x$, we have $\lim_{\Delta x \to 0} c(x, \Delta x) = x$. Since f is continuous at x,

$$\lim_{\Delta x \to 0+} f\left(c\left(x,\Delta x\right)\right) = \lim_{\Delta x \to 0} f\left(c\left(x,\Delta x\right)\right) = f\left(\lim_{\Delta x \to 0} c\left(x,\Delta x\right)\right) = f(x).$$

Therefore,

$$F'_{+}(x) = \lim_{\Delta x \to 0+} f\left(c\left(x, \Delta x\right)\right) = f\left(x\right).$$

If $\Delta x < 0$

$$\frac{F(x+\Delta x)-F(x)}{\Delta x} = \frac{1}{\Delta x}\int_{x}^{x+\Delta x} f(t)\,dt = \frac{1}{(-\Delta x)}\left(-\int_{x}^{x+\Delta x} f(t)\,dt\right) = \frac{1}{(-\Delta x)}\int_{x+\Delta x}^{x} f(t)\,dt.$$

The final expression is the mean value of f on the interval $[x + \Delta x, x]$. By the Mean Value Theorem for Integrals, there exists $c(x, \Delta x) \in [x+\Delta x, x]$ such that

$$\frac{1}{(-\Delta x)}\int_{x+\Delta x}^{x} f(t)\,dt = f\left(c\left(x, \Delta x\right)\right).$$

Therefore,

$$F'_{-}(x) = \lim_{\Delta x \to 0-} \frac{F(x+\Delta x)-F(x)}{\Delta x} = \lim_{\Delta x \to 0-} f\left(c\left(x, \Delta x\right)\right) = f\left(\lim_{\Delta x \to 0-} c\left(x, \Delta x\right)\right) = f\left(x\right),$$

since $c(x, \Delta x)$ is between $x + \Delta x$ and x, and f is continuous at x.

Thus,

$$F'(x) = F'_{+}(x) = F'_{-}(x) = f(x).$$

■

Example 4 (The function erf) Set

$$F(x) = \int_{0}^{x} \frac{2}{\sqrt{\pi}} e^{-t^2}\,dt$$

a) Determine $F'(x)$.
b) Interpret $F(x)$ in terms of area.

Solution

a) By the second part of the Fundamental Theorem of Calculus,

$$F'(x) = \frac{d}{dx}\int_{0}^{x} \frac{2}{\sqrt{\pi}} e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

at each $x \in \mathbb{R}$, since

$$f(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$$

is continuous on $\mathbb{R}$.

b) If $x > 0$, $F(x)$ is the area between the graph of f and the interval $[0, x]$, as illustrated in Figure 12.

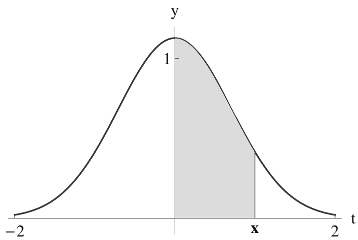


Figure 12: $F(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2} dt$

If $x < 0$,

$$F(x) = \int_0^x f(t)\,dt = -\int_x^0 f(t)\,dt,$$

so that $F(x)$ is $(-1) \times$ (the area between the graph of f and the interval $[x, 0]$), as illustrated in Figure 13.

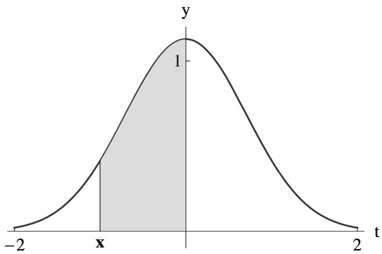


Figure 13: $F(x) = -\int_x^0 \frac{2}{\sqrt{\pi}} e^{-t^2} dt$

The function F is a built-in function in computer algebra systems such as Maple or Mathematica, since it occurs in many statistical applications, and is referred to as **the error function erf.** Figure 14 shows the graph of F. □

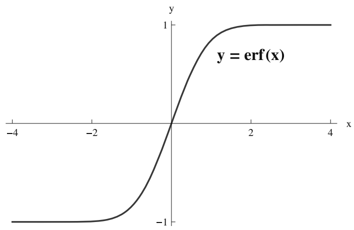


Figure 14: $y = \text{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2}\, dt$

Example 5 (The natural logarithm defined as an integral)

If $x > 0$, we have

$$\int_1^x \frac{1}{t}\, dt = \ln(t)\Big|_1^x = \ln(x) - \ln(1) = \ln(x),$$

since

$$\frac{d}{dt}\ln(t) = \frac{1}{t},\ t > 0.$$

Thus, $\ln(x)$ is the area between the graph of $y = 1/t$ and the interval $[1, x]$ if $x > 1$, as illustrated in Figure 15.

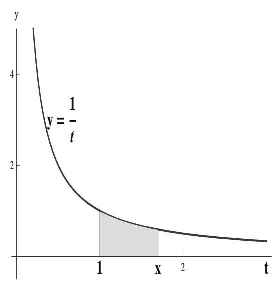


Figure 15: $\ln(x) = \int_1^x \frac{1}{t}\, dt$

If $0 < x < 1$,
so that $\ln(x)$ is $(-1) \times$ (the area between the graph of $y = 1/t$ and the interval $[x, 1]$), as illustrated in Figure 16.

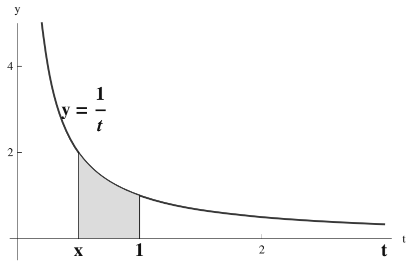


Figure 16: $\ln(x) = -\int_x^1 \frac{1}{t}\,dt$

If we didn't know about the natural logarithm, and needed an antiderivative if $1/x$, we could have set

$$F(x) = \int_1^x \frac{1}{t}dt,\ x > 0.$$

By the second part of the Fundamental Theorem of Calculus, we have

$$F'(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

for each $x > 0$, so that F is an antiderivative of the function defined by $1/x$ on $(0, +\infty)$. Thus, we can introduce the natural logarithm as the function F and define the natural exponential function as its inverse. This approach enables us to derive all the properties of the natural logarithm and the natural exponential function rigorously. □

Example 6 The sine integral function **Si** is defined by the expression

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$$

Determine $\text{Si}'(x)$.

Solution

Since

$$\lim_{t\to 0}\frac{\sin(t)}{t} = 1,$$

if we set

$$f(t)\begin{cases} \dfrac{\sin(t)}{t} & \text{if} \quad t \neq 0, \\ 1 & \text{if} \quad t = 0, \end{cases}$$

then f is continuous on the entire number line. We can interpret the integral

$$\int_0^x \frac{\sin(t)}{t}\,dt$$

as

$$\int_0^x f(t)\,dt.$$

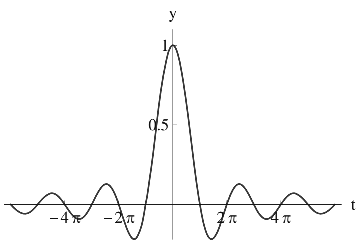


Figure 17: $y = \dfrac{\sin(t)}{t}$

Thus, the second part of the Fundamental Theorem of Calculus is applicable:

$$\frac{d}{dx}\mathrm{Si}(x) = \frac{d}{dx}\int_0^x f(t)\,dt = \begin{cases} \dfrac{\sin(x)}{x} & \text{if} \quad x = 0, \\ 1 & \text{if} \quad x = 0. \end{cases}$$

Figure 17 shows the graph of f and Figure 18 shows the graph of the sine integral function Si. □

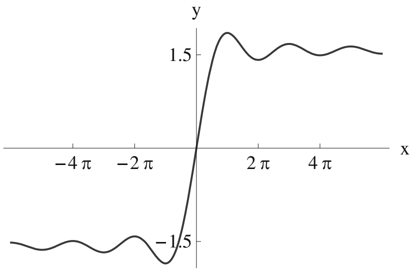


Figure 18: The sine integral function

Example 7 Set

$$F(x) = \int_0^x \frac{t^2 - 4}{t^2 + 1} dt.$$

a) Determine $F'(x)$.

b) Determine the intervals on which F is increasing/decreasing,

Solution

a) The integrand is continuous on the entire number line, since it is a rational function and $t^2 + 1 \neq 0$ for any $t \in \mathbb{R}$. Figure 19 shows the graph of the integrand.

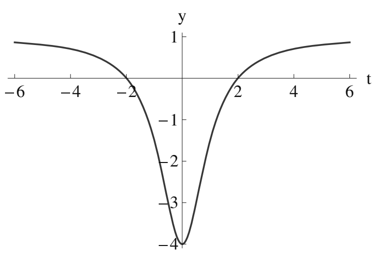


Figure 19: $y = \dfrac{t^2 - 4}{t^2 + 1}$

By the second part of the Fundamental Theorem of Calculus,

$$F'(x) = \frac{d}{dx} \int_0^x \frac{t^2 - 4}{t^2 + 1} dt = \frac{x^2 - 4}{x^2 + 1} \text{ for each } x \in \mathbb{R}.$$

b) We will make of the derivative test for monotonicity. We have

$$F'(x) = 0 \Leftrightarrow \frac{x^2 - 4}{x^2 + 1} = 0 \Leftrightarrow x = \pm 2.$$

We also have $F'(x) > 0$ if $x < -2$, $F'(x) < 0$ if $-2 < x < 2$ and $F'(x) > 0$ if $x > 2$. Therefore, F is increasing on $(-\infty, 2]$, decreasing on $[-2, 2]$ and increasing on $[2, +\infty)$.

In the next chapter we will introduce new special functions and we will be able to express $F(x)$ in terms of one of these functions. In the mean time, you can make use of your computational utility to obtain approximates values for F (for example, you can use midpoint sums), and plot a graph of F. Figure 20 shows such a graph.

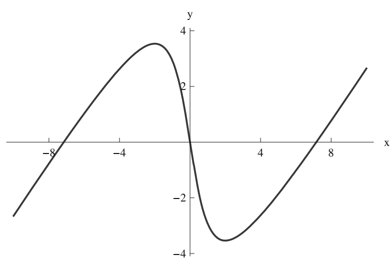


Figure 20: $F(x) = \int_0^x \frac{t^2 - 4}{t^2 + 1} dt$

Incidentally,

$$F(2) = \int_0^2 \frac{t^2 - 4}{t^2 + 1} dt \cong -3.535\,74 \text{ and } F(-2) = \int_0^{-2} \frac{t^2 - 4}{t^2 + 1} dt \cong 3.535\,74$$

□

Now that we have established the second part of the Fundamental Theorem of Calculus, let us display both parts of the Theorem in a symmetric fashion (the restrictions on the functions have been stated earlier):

The Fundamental Theorem of Calculus

1. $$\int_a^x \frac{df(t)}{dt} dt = f(x) - f(a).$$

2.
$$\frac{d}{dx}\int_{\alpha}^{x} f(t)\,dt = f(x).$$

It is worthwhile to repeat the meaning of the Fundamental Theorem: **The first part of the theorem states that the integral of the derivative of a function on an interval is the difference between the values of the function at the endpoints. The second part of the theorem states that the derivative of the function**

$$\int_{a}^{x} f(t)\,dt$$

is the value of the integrand at the upper limit. We can say that differentiation and integration are reverse operations in the precise sense of the Fundamental Theorem.

We may refer to either part of the Fundamental Theorem of Calculus simply as "the Fundamental Theorem of Calculus".

CHAPTER 5

The Indefinite and Definite Integrals of Linear Combinations of Functions

In Chapter 3 we introduced the first part of the Fundamental Theorem of Calculus that enabled us to compute the exact value of an integral once we identified an antiderivative of the integrand. We displayed a short of list of the indefinite integrals of some basic functions. In this chapter will discuss the indefinite and definite integrals of linear combinations of functions, and expand the scope of our short list considerably. We will also discuss the calculation of the area of a region between the graphs of functions

The Linearity of Indefinite and Definite Integrals

The rules for the indefinite integrals of constant multiples and sums of functions follow from the corresponding rules for differentiation:

Proposition 1

1 **(THE CONSTANT MULTIPLE RULE FOR INDEFINITE INTEGRALS) If *c* is a constant,**

$$\int cf(x)\,dx = c\int f(x)\,dx.$$

2 **(THE SUM RULE FOR INDEFINITE INTEGRALS)**

$$\int \left(f(x)+g(x)\right)dx = \int f(x)\,dx + \int g(x)\,dx.$$

Remark 1 Since the indefinite integral of a function represents any antiderivative of the function, and any two antiderivatives of the same function can differ by a constant, **it should be understood that arbitrary con-**

stants can be added to either side of an equality that involves indefinite integrals. ◊

The Proof of Proposition 1

1. Assume that F is an antiderivative of f. Thus,

$$\frac{d}{dx}F(x) = f(x)$$

for each x in an interval J, so that

$$\int f(x)\,dx = F(x).$$

By the constant multiple rule for differentiation,

$$\frac{d}{dx}\left(cF(x)\right) = c\frac{d}{dx}F(x) = cf(x)$$

for each $x \in J$. Therefore,

$$\int cf(x)\,dx = cF(x) = c\int f(x)\,dx.$$

2. Assume that

$$\frac{d}{dx}F(x) = f(x) \text{ and } \frac{d}{dx}G(x) = g(x)$$

for each x in an interval J, so that

$$\int f(x)\,dx = F(x) \text{ and } \int g(x)\,dx = G(x).$$

By the sum rule for differentiation,

$$\frac{d}{dx}\left(F(x)+G(x)\right) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x) = f(x) + g(x)$$

for each $x \in J$. Therefore,

$$\int \left(f(x)+g(x)\right)dx = F(x)+G(x) = \int f(x)\,dx + \int g(x)\,dx.$$

■

Example 1

a) Determine

$$\int 4\cos(x)\,dx.$$

b) Evaluate

$$\int_{\pi/6}^{\pi/4} 4\cos(x)\,dx.$$

Solution

a) We have

$$\int \cos(x)\,dx = \sin(x)$$

(on the entire number line). By the constant multiple rule for indefinite integrals,

$$\int 4\cos(x)\,dx = 4\int \cos(x)\,dx = 4\sin(x).$$

As in Remark 1, it should be understood that an arbitrary constant C can be added to 4 sin (x). If we wish to emphasize this, we may write

$$\int 4\cos(x)\,dx = 4\sin(x) + C.$$

b) The constant in the expression for the indefinite integral is not relevant to the evaluation of the definite integral. Any antiderivative will do for the evaluation of the definite integral with the help of the Fundamental Theorem of Calculus. Thus,

$$\int_{\pi/6}^{\pi/4} 4\cos(x)\,dx = 4\sin(x)\Big|_{\pi/6}^{\pi/4} = 4\sin\left(\frac{\pi}{4}\right) - 4\sin\left(\frac{\pi}{6}\right)$$
$$= 4\left(\frac{\sqrt{2}}{2}\right) - 4\left(\frac{1}{2}\right) = 2\sqrt{2} - 2.$$

□

Example 2 Evaluate

$$\int_0^2 \left(x^2 + 1\right) dx.$$

Solution

By the sum rule for indefinite integrals and the reverse power rule,

$$\int (x^2 + 1)\,dx = \int x^2 dx + \int 1 dx = \frac{1}{3}x^3 + x + C,$$

where C is an arbitrary constant. The constant can be ignored for the evaluation of the definite integral. By the Fundamental Theorem of Calculus,

$$\int (x^2 + 1)\,dx = \frac{1}{3}x^3 + x\Big|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}.$$

□

We can combine parts 1 and 2 of Proposition 1 and obtain the rule for indefinite integrals that corresponds to the linearity of differentiation:

Theorem 1 (THE LINEARITY OF INDEFINITE INTEGRALS) Assume that c_1 and c_2 are constants. Then

$$\int (c_1 f(x) + c_2 g(x))\, dx = c_1 \int f(x)\,dx + c_2 \int g(x)\,dx.$$

Proof

By the sum rule for indefinite integrals,

$$\int (c_1 f(x) + c_2 g(x))\, dx = \int c_1 f(x)\,dx + \int c_2 g(x)\,dx.$$

By the constant multiple rule for indefinite integrals,

$$\int c_1 f(x)\,dx + \int c_2 g(x)\, dx = c_1 \int f(x)\,dx + c_2 \int g(x)\,dx.$$

Therefore,

$$\int (c_1 f(x) + c_2 g(x))\, dx = c_1 \int f(x)\,dx + c_2 \int g(x)\,dx.$$

□

Example 3 Let

$$f(x) = (x+2)(x-1)(x-3) = x^3 - 2x^2 - 5x + 6.$$

a) Determine

$$\int f(x)\,dx.$$

b) Sketch the region G between the graph of f and the interval $[-1, 3]$. Compute the signed area of G and the area of G.

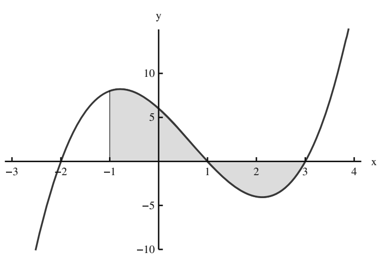
Solution

a) By the linearity of indefinite integrals and the reverse power rule,

$$\begin{aligned}\int (x^3 - 2x^2 - 5x + 6)\,dx &= \int x^3 dx - 2\int x^2 dx - 5\int x dx + 6\int 1 dx \\ &= \frac{x^4}{4} - 2\left(\frac{x^3}{3}\right) - 5\left(\frac{x^2}{2}\right) + 6x + C \\ &= \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x + C,\end{aligned}$$

where C is an arbitrary constant.

b) Figure 1 shows the region G.



The signed area of G is

$$\begin{aligned}\int_{-1}^{3} f(x)\,dx &= \int f(x)\,dx\Big|_{-1}^{3} \\ &= \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\Big|_{-1}^{3} \\ &= -\frac{9}{4} - \left(-\frac{91}{12}\right) = -\frac{9}{4} + \frac{91}{12} = \frac{16}{3}.\end{aligned}$$

Since $f(x) > 0$ if $-1 < x < 1$ and $f(x) < 0$ if $1 < x < 3$, the area of G is

$$\int_{-1}^{1} f(x)\,dx - \int_{1}^{3} f(x)\,dx = \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\Big|_{-1}^{1}\right) - \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\Big|_{1}^{3}\right)$$

$$= \frac{32}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3} + \frac{16}{3} = 16.$$

□

A polynomial is a linear combination of a constant function and functions defined by positive-integer powers of x. Therefore we can determine the indefinite integral of any polynomial, as in Example 3.

Example 4

a) Determine

$$\int \left(\sin(x) + \frac{1}{3}\sin(3x)\right) dx$$

b) Compute

$$\int_{\pi/3}^{\pi} \left(\sin(x) + \frac{1}{3}\sin(3x)\right) dx.$$

Solution

a) The formula

$$\int \sin(\omega x)\,dx = -\frac{1}{\omega}\cos(\omega x) + C,$$

where ω is a nonzero constant and C is an arbitrary constant is on the short list of Section 5.3. With the help of the linearity of indefinite integrals,

$$\int \left(\sin(x) + \frac{1}{3}\sin(3x)\right) dx = \int \sin(x)\,dx + \frac{1}{3}\int \sin(3x)\,dx$$

$$= -\cos(x) + \frac{1}{3}\left(-\frac{1}{3}\cos(3x)\right)$$

$$= -\cos(x) - \frac{1}{9}\cos(3x) + C,$$

where C is an arbitrary constant.

b) By the result of part a) and the Fundamental Theorem of Calculus,

$$\int_{\pi/3}^{\pi}\left(\sin(x)+\frac{1}{3}\sin(3x)\right)dx=-\cos(x)-\frac{1}{9}\cos(3x)\Big|_{\pi/3}^{\pi}$$
$$=\left(-\cos(\pi)-\frac{1}{9}\cos(3\pi)\right)-\left(-\cos\left(\frac{\pi}{3}\right)-\frac{1}{9}\cos(\pi)\right)$$
$$=\frac{10}{9}+\frac{7}{18}=\frac{3}{2}.$$

Figure 2 shows the region G between the graph of f and the interval $[\pi/3, \pi]$. Since $f(x) > 0$ if $\pi/3 < x < \pi$, the area of G is

$$\int_{\pi/3}^{\pi} f(x)\,dx=\frac{3}{2}.$$

□

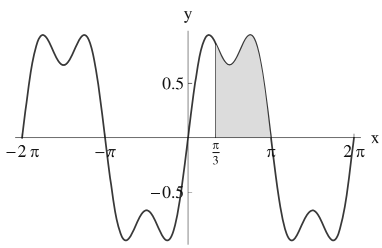


Figure 2

Recall that **a trigonometric polynomial** can be expressed as a linear combination of a constant function and functions defined by sin (nx) and cos (nx), where n is a positive integer. We can determine the indefinite integral of a trigonometric polynomial as in Example 4.

As in the above examples, the linearity of indefinite integrals enables us to calculate the definite integrals of linear combinations of functions whose indefinite integrals are known. Nevertheless, we will need to refer to **the linearity of definite integrals** as well.

Proposition 2 Assume that f and g are integrable on the interval $[a, b]$ and c is a constant. Then,

1 (THE CONSTANT MULTIPLE RULE FOR DEFINITE INTEGRALS)

$$\int_a^b cf(x)\,dx=c\int_a^b f(x)\,dx.$$

2 (THE SUM RULE FOR DEFINITE INTEGRALS)

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

As in the case of indefinite integrals, Proposition 2 leads to the linearity of the definite integral:

Theorem 2 (THE LINEARITY OF DEFINITE INTEGRALS) Assume that f and g are integrable on the interval $[a, b]$, and c_1, c_2 are constants. Then

$$\int_a^b (c_1 f(x)+c_2 g(x))\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx.$$

We will leave the rigorous proof of Proposition 2 to a course in advanced calculus. Let's discuss the plausibility of the statements of Proposition 2 under the assumption that f and g are continuous on $[a, b]$:

Let $m_n(f)$ and $m_n(g)$ denote the midpoint sums for f and g, respectively, corresponding to the partitioning of $[a, b]$ to n subintervals of equal length. We have

$$\lim_{n\to\infty} m_n(f) = \int_a^b f(x)\,dx \text{ and } \lim_{n\to\infty} m_n(g) = \int_a^b g(x)\,dx.$$

Therefore,

$$\lim_{n\to\infty}(m_n(f)+m_n(g)) = \lim_{n\to\infty} m_n(f) + \lim_{n\to\infty} m_n(g) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

You can confirm that $m_n(f) + m_n(g)$ is a midpoint sum for $f + g$, corresponding to the partitioning of $[a, b]$ to n subintervals of equal length, and $f + g$ is continuous on $[a, b]$. Therefore,

$$\lim_{n\to\infty}(m_n(f)+m_n(g)) = \int_a^b (f(x)+g(x))\,dx.$$

Thus,

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Similarly, $cm_n(f)$ is a midpoint sum for cf corresponding to the partitioning of $[a, b]$ to n subintervals of equal length, so that

$$\lim_{n\to\infty} cm_n(f) = \int_a^b cf(x)\,dx.$$

Since

$$\lim_{n\to\infty} cm_n(f) = c\lim_{n\to\infty} m_n(f) = c\int_a^b f(x)\,dx,$$

we have

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

■

Unlike Theorem 1, Theorem 2 is applicable even if we are not able to recognize the antiderivatives of f and g, as in the following example.

Example 5 It is known that

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{6} \text{ and } \int_0^1 \sqrt{4-x^2}\,dx = \frac{\sqrt{3}}{2} + \frac{\pi}{3}.$$

Determine

$$\int_0^1 \left(\frac{2}{\sqrt{4-x^2}} - 3\sqrt{4-x^2}\right)dx.$$

Solution

By the linearity of the definite integral,

$$\begin{aligned}\int_0^1 \left(\frac{2}{\sqrt{4-x^2}} - 3\sqrt{4-x^2}\right)dx &= 2\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx - 3\int_0^1 \sqrt{4-x^2}\,dx \\ &= 2\left(\frac{\pi}{6}\right) - 3\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) \\ &= -\frac{2\pi}{3} - \frac{3\sqrt{3}}{2}.\end{aligned}$$

□

The Area of a Region between the Graphs of Functions

Thanks to the linearity of integration, we can calculate the area of a region between the graphs of two functions. In order to be specific, let's assume that f and g are continuous on $[a, b]$, $f(c) = g(c)$, where $a < c < b$, $f(x) > g(x)$ if $a \le x \le c$, and $g(x) > f(x)$ if $c < x \le b$. Figure 3 illustrates such a case.

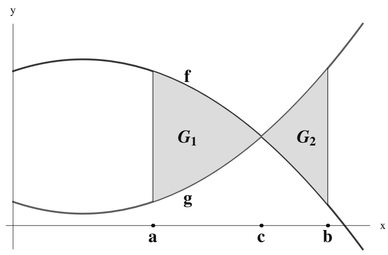


Figure 3

We would like to calculate the area of the region G between the graph of f, the graph of g, the line $x = a$, and the line $x = b$. With reference to Figure 3, the area of G is the sum of the areas of G_1 and G_2. The area of the region G_1 can be obtained by subtracting the area of the region between the graph of g and the interval $[a, c]$ from the area of the region between the graph of f and $[a, c]$. Thus, the area of G_1 is

$$\int_a^c f(x)\,dx - \int_a^c g(x)\,dx = \int_a^c \big(f(x) - g(x)\big)\,dx.$$

Similarly, the area of the region G_2 is

$$\int_c^b \big(g(x) - f(x)\big)dx.$$

Thus, the area of G is

$$\int_a^c \big(f(x) - g(x)\big)\,dx + \int_c^b \big(g(x) - f(x)\big)\,dx.$$

Note that $|f(x) - g(x)| = f(x) - g(x)$ if $a \le x \le c$, since $f(x) \ge g(x)$ in this case. If $c \le x \le b$, then $|f(x) - g(x)| = -(f(x) - g(x)) = g(x) - f(x)$, since $g(x) \ge f(x)$ for each $x \in [c, b]$. Therefore,

$$\int_a^c (f(x)-g(x))\,dx+\int_c^b (g(x)-f(x))\,dx=\int_a^c \left|f(x)-g(x)\right|dx+\int_c^b |f(x)-g(x)|\,dx$$
$$=\int_a^b |f(x)-g(x)|\,dx.$$

Thus, the area of G can be expressed as

$$\int_a^b |f(x)-g(x)|\,dx.$$

This fact is true in the general case: **If f and g are continuous on [a, b], the area of the region between the graph of f, the graph of g, the line $x = a$ and $x = b$ is**

$$\int_a^b |f(x)-g(x)|\,dx.$$

As a special case, we can express **the area of the region between the graph of a function f and an interval [a, b] as**

$$\int_a^b |f(x)|\,dx.$$

($g = 0$).

Remark 2 We can arrive at the expression for the area of a region between the graphs of functions by going back to the definition of the integral. Thus, assume that the graphs of f and g are as in Figure 4.

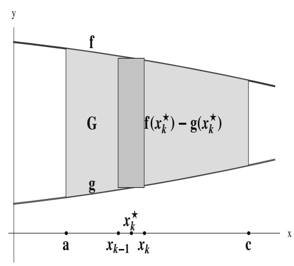


Figure 4

Let $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ be a partition of $[a, c]$. If $\Delta x_k = x_k - x_{k-1}$, and $||P|| = \max_k \Delta x_k$ is small, we can approximate the area of the slice of the region G between the lines $x = x_{k-1}$ and $x = x_k$ by the area of a rectangle whose dimensions are Δx_k and $f(x_k^*) - g(x_k^*)$, where x_k^* is an arbitrary point between x_{k-1} and x_k. The area of such a rectangle is

$$\left(f\left(x_k^*\right)-g\left(x_k^*\right)\right)\Delta x_k,$$

as illustrated in Figure 4. Thus, the sum

$$\sum_{k=1}^{n}\left(f\left(x_k^*\right)-g\left(x_k^*\right)\right)\Delta x_k$$

approximates the area of G if $\max_k \Delta x_k$ is small. But this is a Riemann sum that approximates

$$\int_a^c \left(f(x)-g(x)\right) dx.$$

Therefore, we have reached the same expression for the area of G as before. ◊

Example 6 Let $f(x) = x^2 - 2x - 1$ and $g(x) = -x^2 + 2x + 5$.

a) Sketch the region G between the graph of f, the graph of g, the line $x = 1$ and the line $x = 5$.

b) Calculate the area of G.

Solution

a) Figure 5 shows the region G.

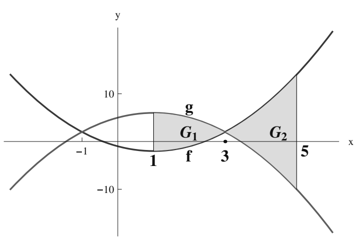


Figure 5

b) In order to determine the x-coordinates of the points at which the graphs of f and g intersect, we need to find the solutions of the equation $f(x) = g(x)$:

$$x^2-2x-1=-x^2+2x+5 \Leftrightarrow 2x^2-4x-6=0$$
$$\Leftrightarrow x-1 \text{ or } x=3.$$

With reference to Figure 5, the area of G is the sum of the areas of G_1 and G_2.

$$\text{The area of } G_1 = \int_1^3 (g(x) - f(x))dx = \int_1^3 \left(-2x^2 + 4x + 6\right) dx$$

$$= -\frac{2}{3}x^3 + 2x^2 + 6x\Big|_{x=1}^{3} = \frac{32}{3}.$$

$$\text{The area of } G_2 = \int_3^5 (f(x) - g(x))\, dx = \int_3^5 \left(2x^2 - 4x - 6\right) dx$$

$$= \frac{2}{3}x^3 - 2x^2 - 6x\Big|_3^5 = \frac{64}{3}.$$

Therefore, the area of G is

$$\frac{32}{3} + \frac{64}{3} = 32.$$

□

Example 7 Let $f(x) = -x^2 + 1$ and $g(x) = \sin(x)$.

a) Plot the graphs of f and g and determine approximations to the points x_1 and x_2 such that $x_1 < 0 < x_2$ and the graphs of f and g intersect at the corresponding points, with the help of your calculator.

b) Express the area of the region G between the graphs of f and g and the lines $x = x_1$ and $x = x_2$ as an integral. Determine an approximation to the integral with the help of your calculator.

Solution

a) Figure 6 shows the region G. The picture indicates that the x-coordinates of the points at which the graphs of f and g intersect are approximately −1.5 and 0.5. We have $x_1 \cong -1.409\ 62$ and $x_2 \cong 0.636\ 733$, rounded to 6 significant digits.

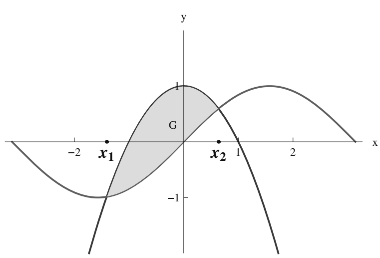


Figure 6

b) Since $f(x) > g(x)$ if $x_1 < x < x_2$, the area of G is

$$\int_{x_1}^{x_2} (f(x) - g(x))dx = \int_{x_1}^{x_2} \left((-x^{2+} + 1\right) - \sin(x))dx$$
$$= -\frac{x^3}{3} + x + \cos(x)\Big|_{x_1}^{x_2} \cong 1.67021.$$

□

Remark 3 Assume that $\boldsymbol{v(t)}$ is the **velocity** at time t of an object in one-dimensional motion, and f is the corresponding **position function.** As we saw in Section 4.3, the **displacement of the object over the time interval** $[\boldsymbol{a, b}]$ is

$$\boldsymbol{f(b) - f(a) = \int_a^b v(t)\,dt.}$$

The **distance** traveled by the object over the same time interval corresponds to the area of the region between the graph of the velocity function on $[a, b]$ and can be expressed as

$$\int_a^b |v(t)|\,dt.$$

The absolute value of the velocity is the **speed** of the object. Thus, **the distance traveled by the object over a time interval is the integral of the speed of the object over that time interval.** ◊

Example 8 Assume that

$$v(t) = 2\sin\left(\frac{t}{4}\right)$$

is the velocity at time t of an object in one-dimensional motion (in centimeters per second).

a) Sketch the graph of v on the interval $[0, 8\pi]$.
b) Determine the displacement of the object over the time interval $[0, 6\pi]$.
c) Determine the distance traveled by the object over the time interval $[0, 6\pi]$.

Solution

a) Figure 7 shows the graph of v on $[0, 8\pi]$.

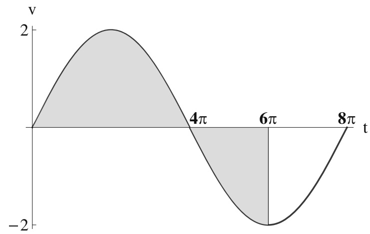


Figure 7

b) The displacement of the object over the time interval $[0, 6\pi]$ is

$$\int_0^{6\pi} v(t)\,dt = \int_0^{6\pi} 2\sin\left(\frac{t}{4}\right)dt = 2\int_0^{6\pi} \sin\left(\frac{t}{4}\right)dt.$$

We have

$$\int \sin\left(\frac{t}{4}\right)dt = -\frac{1}{1/4}\cos\left(\frac{t}{4}\right) + C = -4\cos\left(\frac{t}{4}\right) + C,$$

where C is an arbitrary constant. Therefore,

$$2\int_0^{6\pi} \sin\left(\frac{t}{4}\right)dt = 2\left(-4\cos\left(\frac{t}{4}\right)\Bigg|_0^{6\pi}\right) = 2\left(-4\cos\left(\frac{3\pi}{2}\right) + 4\cos(0)\right) = 8.$$

Thus, the displacement of the object over the time interval $[0, 6\pi]$ is 8 (centimeters).

c) We have $v(t) > 0$ if $0 < t < 4\pi$ and $v(t) < 0$ if $4\pi < t \leq 6\pi$. Therefore the distance traveled by the object over the time interval $[0, 6\pi]$ is

$$\begin{aligned}\int_0^{6\pi} |v(t)|\,dt = \int_0^{4\pi} v(t)\,dt + \int_{4\pi}^{6\pi} -v(t)\,dt &= \int_0^{4\pi} v(t)\,dt - \int_{4\pi}^{6\pi} v(t)\,dt \\ &= \int_0^{4\pi} 2\sin\left(\frac{t}{4}\right)dt - \int_{4\pi}^{6\pi} 2\sin\left(\frac{t}{4}\right)dt \\ &= 16 - (-8) = 24\end{aligned}$$

(check). Graphically, the area of the region between the graph of the velocity function and the interval $[0, 6\pi]$ is 24.□

CHAPTER 6

Using the Substitution Rule for Integrals

The substitution rule for indefinite integrals follows from the chain rule for differentiation. The rule enables us to transform a given antidifferentiation problem to a tractable expression. The definite integral version of the substitution rule is useful in establishing significant general facts.

The Substitution Rule for Indefinite Integrals

Consider the indefinite integral

$$\int \sin\left(x^2\right)2x dx.$$

If we set $u(x) = x^2$, then

$$\frac{du}{dx} = \frac{d}{dx}\left(x^2\right) = 2x.$$

Therefore

$$\int \sin\left(x^2\right)(2x)\,dx = \int \sin(u)\frac{du}{dx}dx.$$

It is tempting to replace the symbol

$$\frac{du}{dx}dx$$

by *du,* and write

$$\int \sin(u)\frac{du}{dx}dx = \int \sin(u)\,du.$$

Assume that the above equality is true. Since we know that

$$\int \sin(u)\,du = -\cos(u) + C,$$

where C is an arbitrary constant, we are led to claim that

$$\int \sin\left(x^2\right) 2x\,dx = -\cos(u) + C = -\cos\left(x^2\right) + C.$$

This is indeed the case, as you can check by differentiating the right-hand side. The procedure that we described above is valid:

Theorem 1 (THE SUBSTITUTION RULE FOR INDEFINITE INTEGRALS) Assume that f is continuous on the interval I, u is a differentiable function on the interval J and $u(x) \in I$ if $x \in J$. Then,

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$$

where $x \in J$.

The expression

$$\int f(u)du$$

denotes a function of u. It should be understood that the above equality is valid, provided that u is replaced by its expression in terms of x. Since the equality involves indefinite integrals, we are entitled to add arbitrary constants to either side.

The Proof of Theorem 1

By the second part of the Fundamental Theorem of Calculus, the continuous function f has an antiderivative F on the interval I. Thus,

$$\frac{d}{du}F(u) = f(u) \Leftrightarrow F(u) = \int f(u)\,du$$

on I. Let's consider the composite function $F \circ u$ on J. By the chain rule,

$$\frac{d}{dx}(F \circ u)(x) = \frac{d}{dx}F(u(x)) = \left(\frac{d}{du}F(u)\bigg|_{u=u(x)}\right)\left(\frac{d}{dx}u(x)\right) = f(u(x))\frac{du}{dx}$$

for each $x \in J$. Therefore

$$\int f\left(u(x)\right)\frac{du}{dx}dx = F\left(u(x)\right)$$

on J. Since

$$F(u) = \int f(u)\,du,$$

we have

$$\int f\left(u(x)\right)\frac{du}{dx}dx = F\left(u(x)\right) = \int f(u)\,du\Big|_{u=u(x)}$$

on J. Therefore,

$$\int f\left(u(x)\right)\frac{du}{dx}dx = \int f(u)\,du,$$

with the understanding that the right-hand side is evaluated at $u(x)$. ■

Example 1 Determine

$$\int \sin^2(x)\cos(x)\,dx.$$

Solution

We set $u(x) = \sin(x)$. Then,

$$\frac{du}{dx} = \frac{d}{dx}\sin(x) = \cos(x).$$

Therefore,

$$\int \sin^2(x)\cos(x)\,dx = \int \left(\sin(x)\right)^2\cos(x)\,dx = \int \left(u(x)\right)^2\frac{du}{dx}dx$$

By the substitution rule and the reverse power rule,

$$\int \left(u(x)\right)^2\frac{du}{dx}dx = \int u^2\,du = \frac{1}{3}u^3 + C,$$

where C is an arbitrary constant. Therefore,

$$\int \sin^2(x)\cos(x)\,dx = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(x) + C.$$

The expression is valid on the entire number line. □

As in the above example, the substitution rule is helpful when it helps us transform the given indefinite integral to a familiar indefinite integral.

Remark 1 It is easy to remember the substitution rule: In the expression

$$\int f(u(x))\frac{du}{dx}\,dx,$$

we can treat

$$\frac{du}{dx}$$

as a "symbolic fraction", carry out "symbolic cancellation" and write

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du.$$

Thus, we can set

$$du = \frac{du}{dx}\,dx$$

when we implement the substitution rule. There is no need to try to attach a mystical meaning to the symbolic manipulation, though: We are merely describing a practical way to remember the substitution rule. Within the present context, the symbol

$$\frac{du}{dx}\,dx$$

does not express the value of the differential $du\,(x, dx)$ that we discussed in Section 2.5, even though the notation is the same. ◊

Example 2 Determine

$$\int x\sqrt{4-x^2}\,dx.$$

Solution

Let's set $u = 4 - x^2$. Then

$$\frac{du}{dx} = \frac{d}{dx}\left(4-x^2\right) = -2x.$$

Therefore,

$$du = \frac{du}{dx}dx = -2xdx.$$

By the substitution rule,

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int\sqrt{4-x^2}\,(-2x)\,dx = -\frac{1}{2}\int\sqrt{u}\,\frac{du}{dx}dx = -\frac{1}{2}\int u^{1/2}du.$$

By the reverse power rule,

$$-\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C = -\frac{1}{3}u^{3/2} + C,$$

where C is an arbitrary constant. Therefore,

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{3}u^{3/2} + C\Big|_{u=4-x^2} = -\frac{1}{3}\left(4-x^2\right)^{3/2} + C.$$

Note that we can take a shortcut by using the formalism

$$du = \frac{du}{dx}dx,$$

and we won't go wrong. Thus

$$du = \frac{d}{dx}\left(4-x^2\right)dx = -2xdx,$$

so that we will replace *xdx* by

$$-\frac{1}{2}du.$$

Therefore,

$$\int x\sqrt{4-x^2}\,dx = \int \sqrt{4-x^2}\,xdx = \int \sqrt{u}\left(-\frac{1}{2}\right)du = -\frac{1}{2}\int u^{1/2}du,$$

as before. □

Remark 2 Note that the implementation of the substitution rule was successful in Examples 1 and 2, since the rule enabled us to transform the given indefinite integral to a constant multiple of

$$\int u^r du,$$

so that we were able to apply the reverse power rule. More generally, if we recognize that the given indefinite integral can be expressed as a constant multiple of

$$\int u^r(x)\frac{du}{dx}dx,$$

the substitution rule leads to $\int u^r\, du$. ◊

Remark 3 In Section 5.3 we noted that

$$\int \sin(\omega x)\,dx = -\frac{1}{\omega}\cos(\omega x) + C \;\; and \int \cos(\omega x)\,dx = \frac{1}{\omega}\sin(\omega x) + C,$$

for any constant $\omega \neq 0$, where C denotes an arbitrary constant. The basic formulas are

$$\int \sin(x)\,dx = -\cos(x) + C \;\; and \int \cos(x)\,dx = \sin(x) + C,$$

since these lead to the more general formulas by the substitution $u = \omega x$. Indeed,

$$u = \omega x \Rightarrow \frac{du}{dx} = \omega.$$

Therefore,

$$\int \sin(\omega x)\,dx = \int \sin(\omega x)\frac{1}{\omega}\omega dx = \frac{1}{\omega}\int \sin(u)\frac{du}{dx}dx = \frac{1}{\omega}\int \sin(u)\,du,$$

by the substitution rule. Thus,

$$\int \sin(\omega x)\,dx = \frac{1}{\omega}\int \sin(u)\,du = -\frac{1}{\omega}\cos(u) + C = -\frac{1}{\omega}\cos(\omega x) + C.$$

The formula that involves cos (ωx) can be obtained in a similar manner. ◊

Example 3

a) Determine

$$\int \tan(x)\,dx.$$

Specify the intervals on which the antidifferentiation formula is valid.

b) Compute

$$\int_{3\pi/4}^{7\pi/6} \tan(x)\,dx.$$

Solution

a) We have

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}dx.$$

Let's set u = cos (x). Then,

$$\frac{du}{dx} = -\sin(x).$$

Therefore,

$$\begin{aligned}\int \frac{\sin(x)}{\cos(x)}dx &= \int \frac{1}{\cos(x)}(\sin(x))\,dx \\ &= \int \frac{1}{u}\left(-\frac{du}{dx}\right)dx \\ &= -\int \frac{1}{u}\frac{du}{dx}dx \\ &= -\int \frac{1}{u}du = -\ln(|u|) + C = -\ln(|\cos(x)|) + C,\end{aligned}$$

where C is an arbitrary constant. Thus,

$$\int \tan(x)\,dx = -\ln(|\cos(x)|) + C.$$

The above expression is valid on any interval of the form

$$\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right), n = 0, \pm 1, \pm 2, \ldots.$$

b) By part a) and the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_{3\pi/4}^{7\pi/6} \tan(x)\,dx &= -\ln(|\cos(x)|)\Big|_{3\pi/4}^{7\pi/6} \\ &= -\ln\left(\left|\cos\left(\frac{7\pi}{6}\right)\right|\right) + \ln\left(\left|\cos\left(\frac{3\pi}{4}\right)\right|\right) \\ &= -\ln\left(\left|-\frac{\sqrt{3}}{2}\right|\right) + \ln\left(\left|-\frac{\sqrt{2}}{2}\right|\right) \\ &= -\ln\left(\frac{\sqrt{3}}{2}\right) + \ln\left(\frac{\sqrt{2}}{2}\right) \\ &= -\frac{1}{2}\ln(3) + \ln(2) + \frac{1}{2}\ln(2) - \ln(2) \\ &= -\frac{1}{2}\ln(3) + \frac{1}{2}\ln(2) \cong -0.202733.\end{aligned}$$

The above integral corresponds to the signed area of the region between the graph of tangent and the interval $[3\pi/4, 7\pi/6]$, as indicated in Figure 1. □

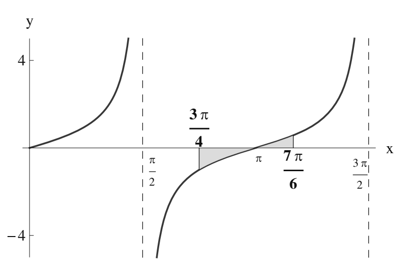


Figure 1

Example 4

a) Determine

$$\int \frac{x}{x^2+1}dx.$$

b) Evaluate

$$\int_2^4 \frac{x}{x^2+1}dx$$

Solution

a) If we set $u = x^2 + 1$, we have

$$\frac{du}{dx} = 2x \Rightarrow x = \frac{1}{2}\frac{du}{dx}.$$

Thus,

$$\int \frac{x}{x^2+1}dx = \int \frac{1}{x^2+1}\left(\frac{1}{2}\frac{du}{dx}\right)dx = \frac{1}{2}\int \frac{1}{u}\frac{du}{dx}dx = \frac{1}{2}\int \frac{1}{u}du.$$

Symbolically,

$$du = \frac{du}{dx}dx = 2xdx \Rightarrow xdx = \frac{1}{2}du,$$

so that

$$\int \frac{x}{x^2+1}dx = \int \frac{1}{x^2+1}xdx = \int \frac{1}{u}\left(\frac{1}{2}\right)du = \frac{1}{2}\int \frac{1}{u}du.$$

Either way,

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln(|u|) + C = \frac{1}{2}\ln\left(\left|x^2+1\right|\right) + C = \frac{1}{2}\ln\left(x^2+1\right) + C,$$

where C is an arbitrary constant.

b) By part a) and the Fundamental Theorem of Calculus,

$$\int_2^4 \frac{x}{x^2+1}dx = \frac{1}{2}\ln\left(x^2+1\right)\Big|_{x=2}^{4} = \frac{1}{2}\ln(17) - \frac{1}{2}\ln(5) \cong 0.611888.$$

The above integral is the area between the graph of

$$y = \frac{x}{x^2 + 1}$$

and the interval [2, 4], as illustrated in Figure 2. □

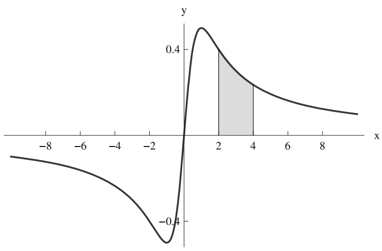


Figure 2

Remark 4 The previous two examples illustrate the appearance of the natural logarithm in many antidifferentiation formulas. Indeed, if we have an indefinite integral that can be expressed as constant multiple of

$$\int \frac{f'(x)}{f(x)} dx,$$

the substitution $u = f(x)$ works:

$$\int \frac{1}{f(x)} f'(x)\, dx = \int \frac{1}{u} \frac{du}{dx} dx = \int \frac{1}{u} du = \ln(|u|) + C = \ln(|f(x)|) + C,$$

where C is an arbitrary constant. ◊

The Substitution Rule for Definite Integrals

An indefinite integral that is determined with the help of the substitution rule can be used to evaluate a definite integral, as in the above examples. There is also a version of the substitution rule which applies directly to definite integrals:

Theorem 2 (THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS) Assume that f is continuous on the interval determined by $u(a)$ and $u(b)$, and that du/dx is continuous on the interval $[a, b]$. Then

$$\int_a^b f(u(x))\frac{du}{dx}dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Proof

The substitution rule for definite integrals is derived in a way that is similar to the derivation of the substitution rule for indefinite integrals. Let F be an antiderivative of f in the interval determined by $u(a)$ and $u(b)$. Thus,

$$\frac{d}{du}F(u) = f(u)$$

if u between $u(a)$ and $u(b)$. By the chain rule,

$$\frac{d}{dx}F(u(x)) = \left(\left.\frac{dF}{du}\right|_{u=u(x)}\right)\frac{du}{dx} = f(u(x))\frac{du}{dx}$$

if $x \in [a, b]$. The first part of the Fundamental Theorem of Calculus implies that

$$F(u(b)) - F(u(a)) = \int_a^b \frac{d}{dx}F(u(x))\,dx = \int_a^b f(u(x))\frac{du}{dx}dx.$$

The first part of the Fundamental Theorem of Calculus also implies that

$$F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} \frac{dF(u)}{du}du = \int_{u(a)}^{u(b)} f(u)\,du.$$

Therefore, we must have

$$\int_a^b f(u(x))\frac{du(x)}{dx}dx = \int_{u(a)}^{u(b)} f(u)\,du,$$

as claimed. ■

Remark 5 (Caution) Even though the substitution rule for definite integrals has an appearance which is similar to the substitution rule for indefinite integrals, Theorem 2 expresses a new rule, since definite and indefinite integrals are different kinds of entities (functions versus numbers). **Also note the change in the limits of integration: The integral on the right-hand side is evaluated from $u(a)$ to $u(b)$, and not from a to b, as in the original integral.** ◊

Example 5 Evaluate

$$\int_0^{\pi/2} \cos^{2/3}(x)\sin(x)\,dx$$

by using the substitution rule for definite integrals.

Solution

We set $u = \cos(x)$ so that

$$du = \frac{du}{dx}dx = \left(\frac{d}{dx}\cos(x)\right)dx = -\sin(x)\,dx.$$

Therefore,

$$\int_0^{\pi/2} \cos^{2/3}(x)\sin(x)\,dx = \int_{u=\cos(0)}^{u=\cos(\pi/2)} u^{2/3}\left(-\frac{du}{dx}\right)dx$$

$$= -\int_1^0 u^{2/3}\,du = \int_0^1 u^{2/3}\,du = \left.\frac{u^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right|_0^1 = \left.\frac{3}{5}u^{5/3}\right|_0^1 = \frac{3}{5}.$$

Figure 3 shows the graph of

$$f(x) = \cos^{2/3}(x)\sin(x).$$

The integral that we calculated is the area of the shaded region. □

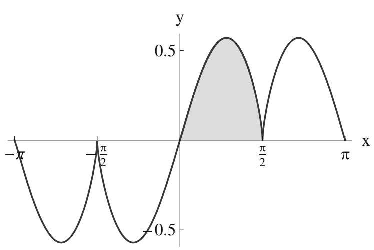


Figure 3

Example 6 Evaluate

$$\int_{\sqrt{\ln(2)}}^{\sqrt{\ln(3)}} e^{-x^2} x dx$$

by using the substitution rule for definite integrals.

Solution

We set $u = -x^2$, so that

$$\frac{du}{dx} = -2x,\ u\left(\sqrt{\ln(2)}\right) = -\ln(2) \text{ and } u\left(\sqrt{\ln(3)}\right) = -\ln(3).$$

Therefore,

$$\begin{aligned}\int_{\sqrt{\ln(2)}}^{\sqrt{\ln(3)}} e^{-x^2} x dx = \int_{\sqrt{\ln(2)}}^{\sqrt{\ln(3)}} e^{-x^2}\left(-\frac{1}{2}\right)(-2x)\,dx &= -\frac{1}{2}\int_{\sqrt{\ln(2)}}^{\sqrt{\ln(3)}} e^{u}\frac{du}{dx}dx \\ &= -\frac{1}{2}\int_{-\ln(2)}^{-\ln(3)} e^{u} du \\ &= -\frac{1}{2}\int_{-\ln(2)}^{-\ln(3)} e^{u} du \\ &= -\frac{1}{2}\left(e^{u}\Big|_{-\ln(2)}^{-\ln(3)}\right) \\ &= -\frac{1}{2}e^{-\ln(3)} + \frac{1}{2}e^{-\ln(2)} \\ &= -\frac{1}{2}\left(\frac{1}{e^{\ln(3)}}\right) + \frac{1}{2}\left(\frac{1}{e^{\ln(2)}}\right) \\ &= -\frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{12}.\end{aligned}$$

Thus, the area of the region between the graph of $y = e^{-x^2} x$ and the interval $\left[\sqrt{\ln(2)}, \sqrt{\ln(3)}\right]$ that is illustrated in Figure 4 is 1/12. □

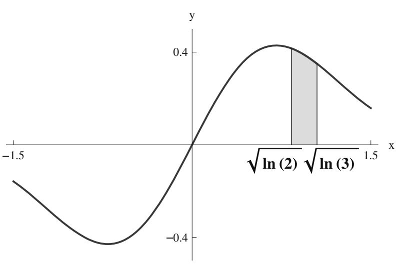


Figure 4

The definite integral version of the substitution rule does not offer an advantage over the indefinite integral version of the rule if

$$\int f(u(x))\frac{du}{dx}\,dx = \int f(u)\,du$$

and

$$\int f(u)\,du$$

can be expressed in terms of familiar functions. On the other hand, the substitution rule for definite integrals leads to useful facts about integrals, as in the following proposition:

Proposition 1

a) If f is even and continuous on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

b) If f is odd and continuous on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Both parts of Proposition 1 are plausible. If f is even, the graph of f is symmetric with respect to the vertical axis. With reference to Figure 5, the area of G_L is the same as the area of G_R.

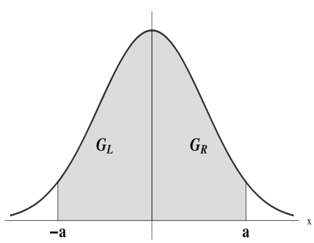


Figure 5

Thus,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = (\text{area of the } G_L) + (\text{area of } G_R)$$
$$= 2\times(\text{area of } G_R) = 2\int_{0}^{a} f(x)\,dx.$$

If f is odd, the graph of f is symmetric with respect to the origin. With reference to Figure 6, the signed area of G_- is $(-1) \times$ the area of G_+.

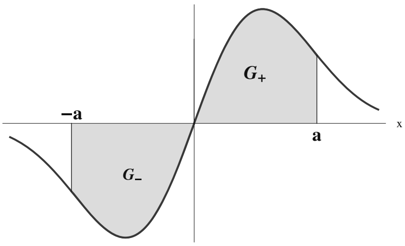


Figure 6

Thus,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = (\text{the signed area of } G_-) + (\text{the area of } G_+) = 0.$$

The Proof of Proposition 1

We will prove part a), and leave the similar proof of part b) as an exercise. Thus, assume that f is even. By the additivity of integrals with respect to intervals,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

Since f is even, we have $f(-x) = f(x)$. Therefore,

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} f(-x)\,dx.$$

Let us apply the substitution rule to this integral by setting $u = -x$. Then, $du/dx = -1$, so that

$$\int_{-a}^{0} f(-x)\,dx = -\int_{-a}^{0} f(u)(-1)\,dx = -\int_{-a}^{0} f(u)\frac{du}{dx}\,dx$$
$$= -\int_{u(-a)}^{u(0)} f(u)\,du = -\int_{a}^{0} f(u)\,du = \int_{0}^{a} f(u)\,du.$$

Thus,

$$\int_{-a}^{0} f(-x)\,dx = \int_{0}^{a} f(u)\,du = \int_{0}^{a} f(x)\,dx$$

(the variable of integration is a dummy variable). Therefore,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$
$$= 2\int_{0}^{a} f(x)dx,$$

as claimed. ■

CHAPTER 7

The Fundamental Theorem of Calculus and the Differential Equation $y' = f'$

In this chapter, we will take another look at the Fundamental Theorem of Calculus within the framework of differential equations and initial-value problems.

In Section 4.6 we saw that **the general solution** of the differential equation $y'(x) = ky(x)$, where k is a constant, is a constant multiple of e^{kx}. Thus, we were able to determine the unique solution of **the initial-value problem**

$$y'(x) = ky(x),\ y(x_0) = y_0,$$

where x_0 and y_0 are given numbers, as

$$y(x) = y_0 e^{k(x - x_0)}.$$

Now we will consider differential equations of the form

$$y'(x) = f(x),$$

where f is a given function, and initial-value problems of the form

$$y'(x) = f(x), y(x_0) = y_0,$$

where x_0 and y_0 are given numbers.

The Differential Equation $y' = f$ and the Fundamental Theorem

We have $y'(x) = f(x)$ for each x in an interval J if and only if y is an antiderivative of f on J. Therefore, we can express y as the indefinite integral of f:

$$y(x) = \int f(x)\,dx.$$

We will refer to

$$\int f(x)\,dx$$

as **the general solution of the differential equation $y'(x) = f(x)$.** The indefinite integral involves an arbitrary constant. The value of the constant is determined uniquely if **an initial condition** of the form $y(x_0) = y_0$ is specified, so that the solution of **the initial-value problem,**

$$y'(x) = f(x),\ y(x_0) = y_0$$

is uniquely determined.

Example 1

a) Determine the general solution of the differential equation

$$y'(x) = 2x.$$

b) Determine the solution of the initial-value problem

$$y'(x) = 2x \text{ and } y(2) = 5.$$

Solution

a) We have $y'(x) = 2x$ if and only if

$$y(x) = \int 2x\,dx = x^2 + C,$$

where C is an arbitrary constant. Thus,

$$y(x) = x^2 + C$$

is the general solution of the differential equation $y'(x) = 2x$. Since C is an arbitrary constant, the general solution represents infinitely many functions that differ from x^2 by the addition of a constant. Figure 1 displays the members of this family of functions corresponding to $C = -4$, 1, 4. If (x, y) is on one of the solution curves, the slope of the line that is tangent to that particular solution curve at (x, y) is $2x$. Thus, the tangent lines to the solution curves corresponding to a given x are parallel to each other.

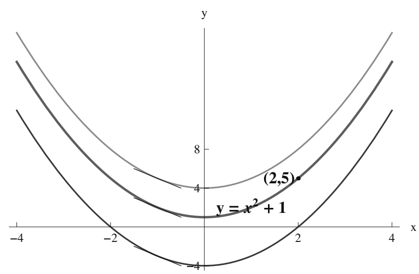


Figure 1

b) Since $y(x) = x^2 + C$ is the general solution of the given differential equation, we have

$$y(2) = 5 \Leftrightarrow 2^2 + C = 5 \Leftrightarrow C = 1.$$

Therefore, the required solution is

$$y(x) = x^2 + 1.$$

The graph of $y = x^2 + 1$ is the only member of the family of curves $y = x^2 + C$ that passes through the point (2, 5). □

Example 2

a) Determine the general solution of the differential equation

$$y'(x) = \sin(4x).$$

b) Determine the solution of the initial-value problems,

$$y'(x) = \sin(4x),\ y(\pi/4) = 2,$$

and

$$y'(x) = \sin(4x),\ y(\pi/4) = -2.$$

Sketch the graphs of the solutions.

Solution

a) We have $y'(x) = \sin(4x)$ if and only if

$$y(x) = \int \sin(4x)\,dx = -\frac{1}{2}\cos(4x) + C,$$

where C is an arbitrary constant. This is the general solution of the differential equation

$$y'(x) = \sin(4x).$$

b) With reference to part a),

$$y\left(\frac{\pi}{4}\right) = -\frac{1}{4}\cos(\pi) + C = \frac{1}{4} + C.$$

Therefore,

$$y(\pi/4) = 2 \Leftrightarrow \frac{1}{4} + C = 2 \Leftrightarrow C = \frac{7}{4}.$$

Thus, the solution of the initial-value problem

$$y'(x) = \sin(4x), y(\pi/4) = 2$$

is

$$F(x) = -\frac{1}{4}\cos(4x) + \frac{7}{4}.$$

Similarly, the solution of the initial-value problem

$$y'(x) = \sin(4x),\ y(\pi/4) = -2$$

is

$$G(x) = -\frac{1}{4}\cos(4x) - \frac{9}{4}.$$

Figure 2 displays the graphs of F and G. Note that

$$F'(x) = G'(x) = \sin(4x),$$

so that the tangent line to the graph of F at the point $(x, F(x))$ is parallel to the tangent line to the graph of G at $(x, G(x))$. ☐

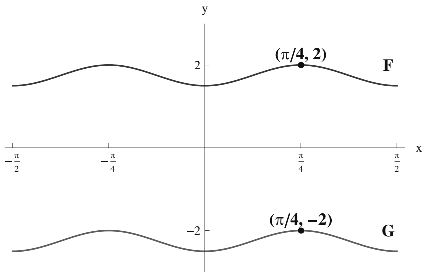


Figure 2

In the above examples, we were able to determine the relevant indefinite integral in terms of familiar functions. This need not be the case. Nevertheless, any continuous function has an antiderivative by the second part of the Fundamental Theorem of Calculus: We have

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

for each $x \in J$ if f is continuous on the interval J and a is a fixed point in J. Therefore, we can express the general solution of the differential equation $y' = f$ on the interval J as

$$y(x) = \int_a^x f(t)\,dt + C,$$

where a is some point in J and C is a constant. If we are given an initial condition of the form $y(x_0) = y_0$, it is convenient to set $a = x_0$. In this case,

$$y(x) = C + \int_{x_0}^x f(t)\,dt,$$

so that

$$y_0 = y(x_0) = C + \int_{x_0}^{x_0} f(t)\,dt = C.$$

Therefore $C = y_0$, and **the unique solution of the initial-value problem**

$$y'(x) = f(x), y(x_0) = y_0$$

can be represented as

$$y(x) = y_0 + \int_{x_0}^{x} f(t)\,dt.$$

In the above expression, we may or may not be able to express the integral in terms of familiar functions. In any case, the values of the solution can be approximated by approximating the integral.

Example 3

a) Express the solution of the initial-value problem,

$$y'(x) = \sin(x^2), y(2) = 3,$$

in terms of an integral.

b) Compute approximations to y (3) and y (4) with the help of the approximate integration facility of your computational utility.

c) Plot the graph of the solution of part a) on the interval [0, 4] with the help of your computational/graphing utility.

Solution

a) We can express the solution as

$$y(x) = 3 + \int_{2}^{x} \sin(t^2)\,dt.$$

b) We have

$$y(3) = 3 + \int_{2}^{3} \sin(t^2)\,dt \cong 2.968\ 79,$$

and

$$y(4) = 3 + \int_{2}^{4} \sin(t^2)\,dt \cong 2.942\ 36.$$

c) Figure 3 shows the graph of the solution on [0, 4]. □

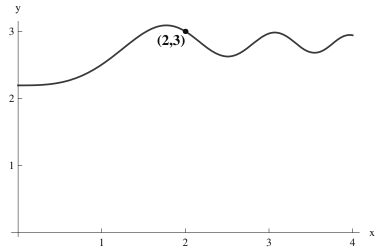


Figure 3

Acceleration, Velocity and Position

Let's consider the relationships between acceleration, velocity and position within the framework of initial-value problems. Assume that $f(t)$ is the **position,** $v(t)$ is the **velocity** and $a(t)$ is the **acceleration** at time t of an object in one-dimensional motion. Velocity is the rate of change of position, and acceleration is the rate of change of velocity:

$$v(t) = \frac{df}{dt} \text{ and } a(t) = \frac{dv}{dt}.$$

When we introduced these concepts initially, we assumed that the position was given, and calculated velocity and acceleration by differentiation. Now we are able to begin with a given acceleration function, and determine the velocity and position functions successively. Thus assume that $a(t)$ is given. The velocity function $v(t)$ is the solution of the differential equation

$$\frac{dv}{dt} = a(t).$$

We have seen that such a differential equation does not have a unique solution. On the other hand, if an initial condition is specified, the solution is uniquely determined. Thus, assume that the velocity at a certain instant t_0 is v_0, so that $v(t_0) = v_0$. We can express the solution of the initial-value problem

$$\frac{dv}{dt} = a(t), v(t_0) = v_0,$$

as

$$\boldsymbol{v}(t)=\boldsymbol{v}_0+\int_{t_0}^{t} a(\tau)\,d\tau.$$

The position function is uniquely determined if the position of the object is specified at some instant. If $f(t_0) = f_0$, the position function is the solution of an initial-value problem

$$\frac{df}{dt}=v(t),\ f(t_0)=f_0.$$

The solution can be expressed as

$$f(t)=f_0+\int_{t_0}^{t} \boldsymbol{v}(\tau)\,d\tau.$$

Example 4 Assume that an object is falling under the influence of gravitational acceleration of 9.8 meters/second/second. The effect of air resistance is neglected. We model the motion as one-dimensional motion so that the number line is vertical, points downward, and the origin coincides with the point at which the object is released. We assume that the object is released from rest. Thus, with the above notation, $a(t) = 9.8$, $v(0) = 0$ and $f(0) = 0$. Determine $v(t)$ and $f(t)$ at any instant t before the object hits the ground.

Solution

We have

$$\frac{dv}{dt}=a(t)=9.8,\ v(0)=0.$$

Therefore,

$$v(t)=\int_0^t a(\tau)\,d\tau=\int_0^t 9.8\,d\tau=9.8t\,(\text{meters/sec.}).$$

We have

$$\frac{df}{dt}=v(t)=9.8t \text{ and } f(0)=0.$$

Therefore,

$$f(t) = \int_0^t v(\tau)\,d\tau = \int_0^t 9.8\tau\,d\tau = \frac{9.8}{2}\tau^2\Big|_0^t = 4.9t^2 \text{ (meters)}.$$

□

Example 5 Assume that the acceleration of an object in simple harmonic motion is 20 cos (6*t*) at the instant *t*. Determine the velocity and the position of the object at the instant *t* if $v(\pi/6) = 0$ and $f(\pi/6) = 2$ (with the notation preceding Example 4).

Solution

We have

$$\frac{dv}{dt} = a(t) = 20\cos(6t) \text{ and } v(\pi/6) = 0.$$

Therefore,

$$\begin{aligned} v(t) = \int_{\pi/6}^t 20\cos(6\tau)\,d\tau &= \frac{10}{3}\sin(6\tau)\Big|_{\pi/6}^t \\ &= \frac{10}{3}\sin(6t) - \frac{10}{3}\sin(\pi) \\ &= \frac{10}{3}\sin(6t). \end{aligned}$$

We have

$$\frac{df}{dt} = v(t) = \frac{10}{3}\sin(6t) \text{ and } f\left(\frac{\pi}{6}\right) = 2.$$

Therefore,

$$\begin{aligned} f(t) &= 2 + \int_{\pi/6}^t \frac{10}{3}\sin(6\tau)\,d\tau \\ &= 2 + \left(-\frac{10}{18}\cos(6\tau)\Big|_{\pi/6}^t\right) \\ &= 2 + \left(-\frac{10}{18}\cos(6t) + \frac{10}{18}\cos(\pi)\right) \\ &= 2 + \left(-\frac{10}{18}\cos(6t) - \frac{10}{18}\right) \\ &= \frac{13}{9} - \frac{5}{9}\cos(6t). \end{aligned}$$

Figure 4 shows the graph of *f.* Note that the motion is periodic with period $\pi/3$. □

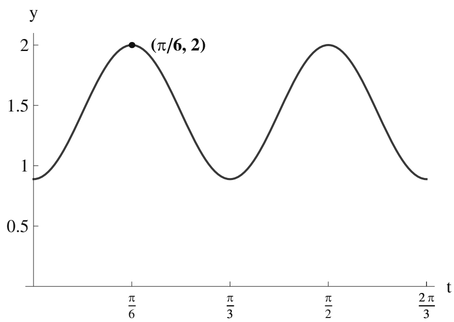


Figure 4

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